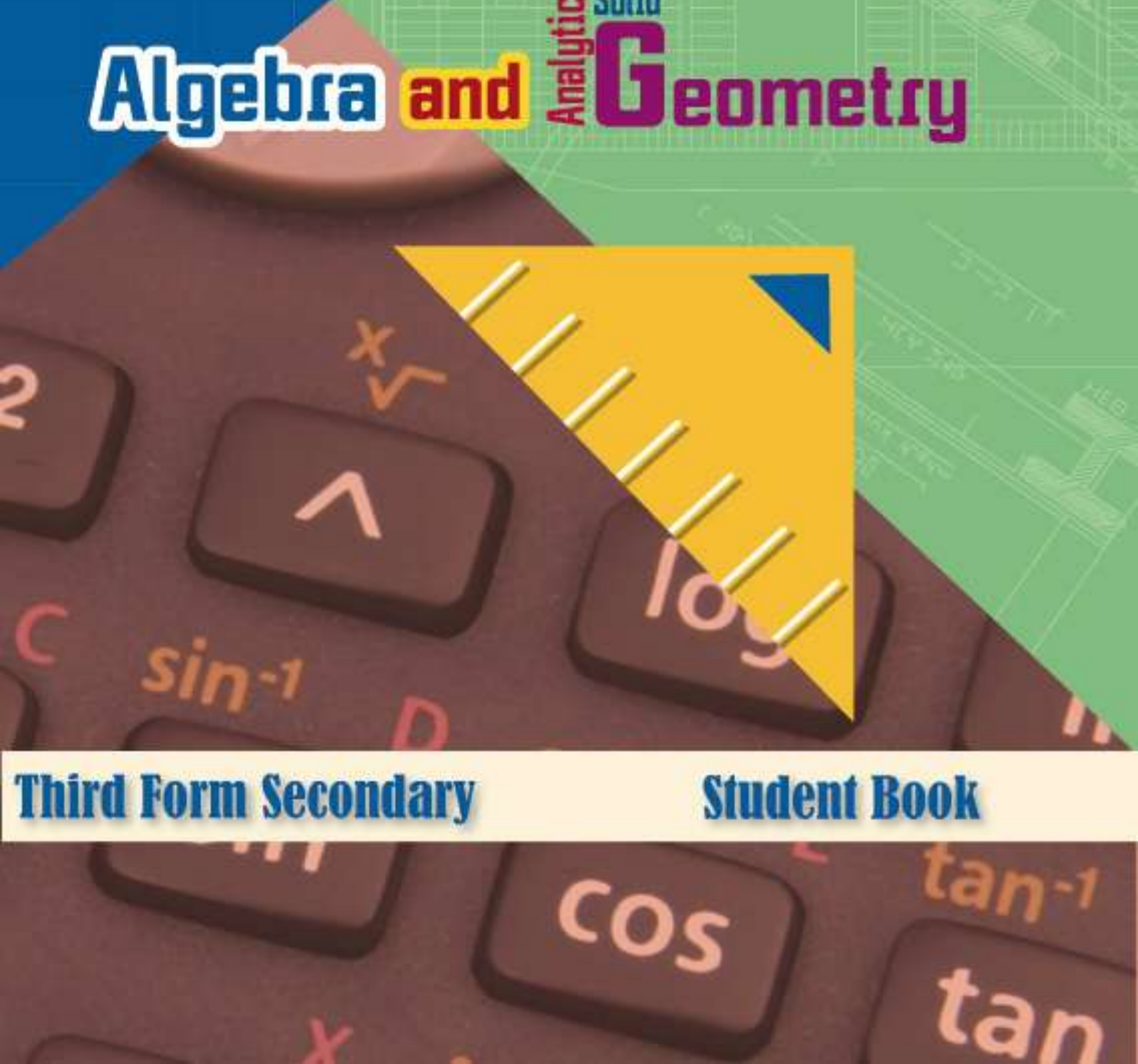
Pure Mathematics

Algebra and Solid Analytic Geometry



Third Form Secondary **Student Book**

Pure Mathematics

Algebra and Solid Analytic Geometry

Third Form Secondary

Student Book

Name: ....

Class: ....

School: ....

Authors

Mr. Kamal Younis Kabsha

Prof.Dr. Mohamed Hossin Fahmy **Mr. Osama Gaber Abdelhafez**

Mr. Ibrahim Abd-Allatif Alsagheir

First eddition	2016/2017
Deposit number	8705 / 2016
I.S.B.N	978 - 977 - 706 - 033 - 2

Introduction

بسم الله الرحمن الرحيم

We are pleased to introduce this book to show the philosophy on which the academic content has been prepared. This philosophy aims at:

1 Developing and integrating the knowledgeable unit in Math, combining the concepts and relating all the school mathematical curricula to each other.

2 Providing learners with the data, concepts, and plans to solve problems.

3 Consolidate the national criteria and the educational levels in Egypt through:

A) Determining what the learner should learn and why.

B) Determining the learning outcomes accurately. Outcomes have seriously focused on the following: learning Math remains an endless objective that the learners do their best to learn it all their lifetime. Learners should like to learn Math. Learners are to be able to work individually or in teamwork. Learners should be active, patient, assiduous and innovative. Learners should finally be able to communicate mathematically.

4 Suggesting new methodologies for teaching through (teacher guide).

5 Suggesting various activities that suit the content to help the learner choose the most proper activities for him/her.

6 Considering Math and the human contributions internationally and nationally and identifying the contributions of the achievements of Arab, Muslim and foreign scientists.

In the light of what previously mentioned, the following details have been considered:

★ This book contains: algebra and analytic solid geometry. The book has been divided into related and integrated units. Each unit has an introduction illustrating the learning outcomes, the unit planning guide, and the related key terms. In addition, the unit is divided into lessons where each lesson shows the objective of learning it through the title You will learn. Each lesson starts with the main idea of the lesson content. It is taken into consideration to introduce the content gradually from easy to hard. The lesson includes some activities, which relate Math to other school subjects and the practical life. These activities suit the students' different abilities, consider the individual differences throughout Discover the error to correct some common mistakes of the students, confirm the principle of working together and integrate with the topic. Furthermore, this book contains some issues related to the surrounding environment and how to deal with.

★ Each lesson contains examples starting gradually from easy to hard and containing various levels of thoughts accompanied with some exercises titled Try to solve. Each lesson ends in Exercises that contain various problems related to the concepts and skills that the students learned through the lesson.

★ Each unit ends in Unit summary containing the concepts and the instructions mentioned and General exams containing various problems related to the concepts and skills, which the student learned through the unit.

★ Each unit ends in an Accumulative test to measure some necessary skills to be gained to fulfill the learning outcome of the unit.

★ The book ends in General tests including some concepts and skills, which the student learned throughout the term.

Last but not least. We wish we had done our best to accomplish this work for the benefits of our dear youngsters and our dearest Egypt.

Contents

First: Algebra

Unit One: Permutations, combinations and Binomial theorem

1 - 1 Fundamental counting principle- permutations - combinations 4
2 - 1 Binomial theorem for integer positive power 13
3 - 1 Finding the term containing x^R in the expansion of binomial 22
4 - 1 Ratio between two consecutive terms of the Binomial expansion 27
Unit summary 30
General exercises 31
Accumulative test 35

Unit Two: Complex number

1 - 2 The trigonometric form of a complex number 38
2 - 2 De Moivre's theorem 50
3 - 2 The cubic roots of unity 55
General exercises 59
Unit summary 61
Accumulative test 63

Unit Three: Determinants and Matrices

1 - 3 Determinants 66
2 - 3 matrices 79
4 - 3 Solving linear equations using the multiplicative inverse of the matrix 86
Unit summary 97
General exercises 99
Accumulative test 102

Algebra

Contents

Second: Analytic solid geometry

Unit One: Geometry and Measurement in two and three dimensions

1 - 1 The 3D orthogonal coordinate system 106

2 - 1 vectors in space. 114

3 - 1 Vector multiplication 123

Unit summary 138

General exercises 144

Accumulative test 146

Unit Two: Straight Lines and planes in space

1 - 2 The equation of a straight line in space. 150

2 - 2 The equation of a plane in space. 160

Unit summary 171

General exercises 173

Accumulative test 174

General Tests & Answer

General tests 176

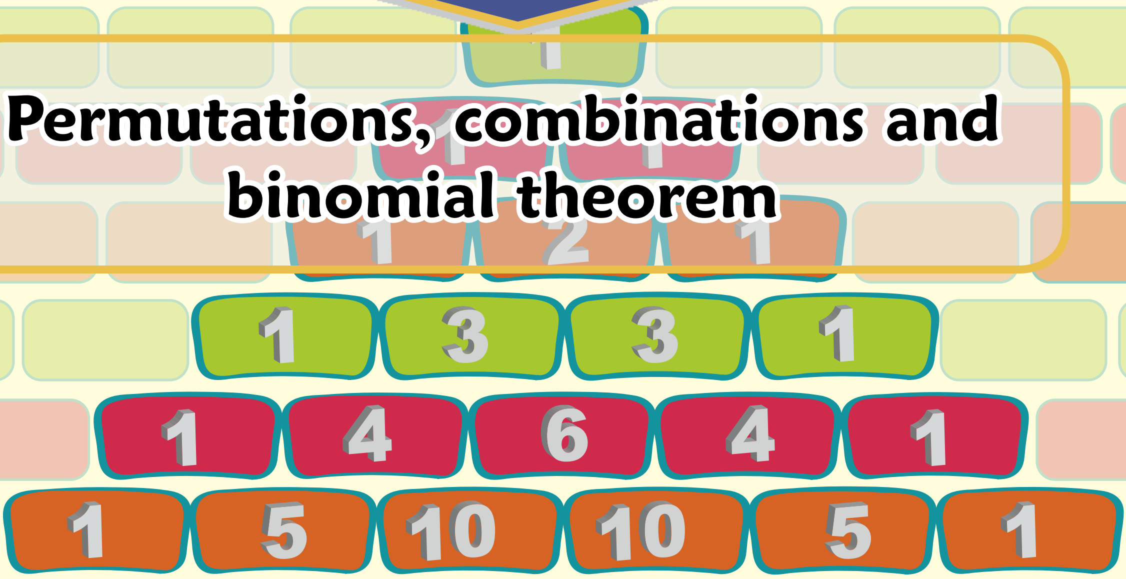
Answers of some exercises 197

Analytic solid geometry

First: Algebra

Unit One

Permutations, combinations and binomial theorem



Unit introduction

Nasser Eldin Eltosy was born (1201 - 1274) in Gahrood near tos in Iran in a family specialized in science and philosophy. He was a student of Kamal Eldin ElMowsly and Moeen Eldin Elmasry, so he studied wisedom, philosophy, astrology and mathematics. He had a great history in calculating the number of possibilities for different phenomena to happen. He also used permutations and combinations. Kardan (1901-1976) also had great interest in calculating the number of possibilities using the fundamental counting principle, which allowed him a great field in computer architecture which is about the design and structure of functional operations for computer. This unit is dealing with the principle of counting and the relation between permutations and combinations and their uses in solving some mathematical problems as well as the binomial theorem and solving some mathematical life applications.

Unit objectives

By the end of this unit and doing all the activities involved , the student should be able to:

- Identify the counting principle (Addition rule)
- Identify the relation between permutations and combinations as methods of counting.
- Deduce rules and results on permutations and combinations.
- Use permutations and combinations in solving mathematical application problems in different fields.
- Identify the binomial theorem with positive integer power
- Deduce the general term in the expansion of the binomial theorem.
- Deduce the ratio between each term and the previous one in the expansion of the binomial theorem.
- Find the coefficient of any term in the expansion of the binomial theorem due to the order of this term.
- Find the coefficient of any power of x in the expansion of $(x + y)^n$
- Find the term free of x in the expansion of $(x + y)^n$
- Find the coefficient of the greatest term in the expansion of the binomial theorem
- Find the middle term in the expansion of the binomial theorem when n is an even number and the two middle terms when n is an odd number.
- Deduce relations between the combinations using the expansion of binomial theorem
- Deduce the relation between pascal's triangle and the coefficients of the expansion of binomial theorem and deduce some patterns using pascal's triangle .
- Solve mathematical life applications on the binomial theorem.

Key Terms

- *Fundamental counting principle*
- *Permutations*
- *Combinations*
- *Binomial Theorem*

Materials

- *Scientific calculator*

Unit Lessons

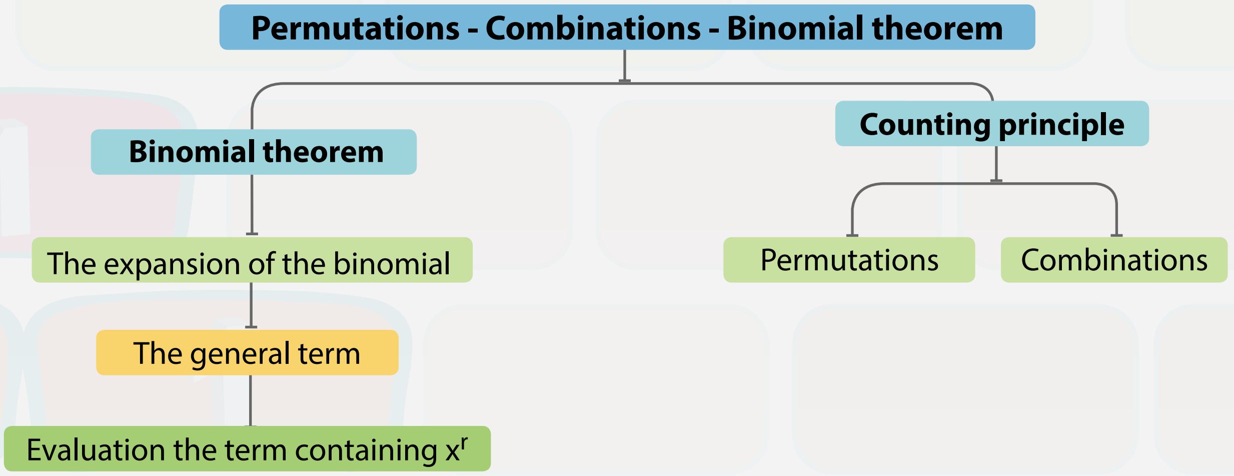
Lesson (1 - 1): Counting principle - Permutations - combinations

Lesson (1 - 2): Bunomial theorem with integer positive power

Lesson (1 - 3): Finding the term containing x^r in the expansion of binomial

Lesson (1 - 4): The ratio between two consecutive terms in the binomial expansion

Unit Chart



Unit One

1 - 1

Fundamental counting principle- permutations - combinations

You will learn

- *Counting principle (Addition rule)*
- *More relation between permutations and combinations.*
- *Applications on using permutations and combinations*

Key terms

- *Permutations*
- *Combinations*
- *Counting precinple*

Materials

- *Scientific calculator*

Preface

First: Counting principle

As you have studied, counting principle (multiplication rule) which states:

If a certain act can be performed in m different ways, and a second act can be performed in n different ways , then the two successive acts can be performed in $(m \times n)$ different ways.

Think and discuss

How many different 3 - digit numbers can be formed from the elements of the set {1, 2, 3, 4, 5}?

since the number formed from (3 digits)

The unit digit can be formed by 5 different ways

The tens digit can be formed by 4 different ways

The hundreds digit can be formed by 3 different ways

Thus. The number of ways to form different 3 digit numbers = $5 \times 4 \times 3 = 60$

Think: How many 3-digit numbers (repeating is allowed) can be formed from the elements of the set {1, 2, 3, 4, 5}?

Learn

Counting principle (Addition rule)

If a certain act can be performed in n different ways, and a second act can be performed in m different ways, then the number of ways to perform the first act or the second act is $(m + n)$ ways.

Example

1. From a class of 9 boys and 6 girls it's required to select a 4-person team of the same gender.

Solution

a. The number of ways to form the team if all members are boys $= {}^9C_4 = 126$

b The number of ways to form the team if its members are girls = $^{6}C_4 = 15$

then , the number of ways to form the team if all of its members from the same gender $= {}^{9}C_4 + {}^{6}C_4 = 126 + 15 = 141$

Try to solve

1. How many ways can 3 persons be selected from 5 men and 4 women in each of the following:
   a The 3 persons are of the same gender? **b** Only 2 persons are of the same gender?

Example

2. A paper exam contains 8 questions. The student should answer 6 of them in the condition that at least two questions are to be from the first four questions. How many ways can the student choose the questions to be answered?

Solution

a The student can choose 2 questions from the first 4 questions and 4 from the rest questions $^{4}C_2 \times {}^{4}C_4 = 6$

b The student can choose 3 questions from the first 4 questions and 3 from the rest questions $^{4}C_3 \times {}^{4}C_3 = 16$

c The student can choose 4 questions from the first 4 questions and 2 from the rest questions $^{4}C_4 \times {}^{4}C_2 = 6$

Number of ways to choose the questions $= {}^{4}C_2 \times {}^{4}C_4 + {}^{4}C_3 \times {}^{4}C_3 + {}^{4}C_4 \times {}^{4}C_2 = 6 + 16 + 6 = 28$

Try to solve

2. A college student at the first year studies 8 subjects and it is not allowed to move on to the next year unless he succeeds in at least 6 of them. How many ways can the student move on to the next year?

Number of ways to select a sample with replacment or without

When you choose r objects out of n objects, we consider the following:

1 - If the selection is with replacement and arrangement, then the number of ways of selections $= n^r$

- Number of ways to form 2-digit number from {1, 2, 3, 4, 5} $= 5^2 = 25$

2 - If the selection is with replacement and without arrangement, then the number of ways of selection $= {}^{n+r-1}C_r$

- Number of ways to distribute 3 identical balls on 4 boxes $= {}^{4+3-1}C_3 = {}^{6}C_3 = 20$

3 - If the selection is without replacement and considers the arrangement, then the number of ways of selection $= {}^{n}P_r$

- Number of ways to park 4 cars in a parking area with 10 places $= {}^{10}P_4 = 10 \times 9 \times 8 \times 7 = 5040$

4 - If the selection is without replacement or arrangement then number of ways of selection $= {}^{n}C_r$

- Number of ways a 5-person team can be chosen from 12 person $= {}^{12}C_5 = 792$

Example

3 A bag contains 12 red balls and 8 white balls, find the number of ways to draw 3 red balls and 2 white balls in each of the following:

a Considering replacement and arrangement.

b Considering arrangement without replacement.

c Without replacement or arrangement.

Solution

a Number of ways = $12^3 \times 8^2 = 110592$

b Number of ways = ${}^{12}P_3 \times {}^{8}P_2 = 73920$

c Number of ways = ${}^{12}C_3 \times {}^{8}C_2 = 6160$

Try to solve

3 In the previous example, find the number of ways to draw 5 balls of the same color in each of the previous cases.

Critical thinking: Find number of ways to park 4 cars next to each other in a parking Area with 10 places to park if:

a The parking area is in a shape of a circle.

b The parking area is in a form of a row.

Second : Permutations:

As you have studied the concept of permutations and known that permutations are each arrangement you can get from several objects by taking them all, or taking some of them also you have studied the following relations:

1) ${}^{n}P_r = n(n-1)(n-2) \times \ldots\ldots \times (n-r+1)$

For each $n \geqslant r$, $n \in Z^+$, $r \in N$

2) ${}^{n}P_n = \underline{|n} = n(n-1)(n-2) \times \ldots\ldots\ldots \times 3 \times 2 \times 1$

3) ${}^{n}P_r = \dfrac{\underline{|n}}{\underline{|n-r}}$ for each $n \geqslant r$, $n \in N$, $r \in N$

4) $\underline{|n} = n\,\underline{|n-1} = n(n-1)\,\underline{|n-2} = \ldots\ldots$

Remember?

${}^{n}P_r \in Z^+$

$\underline{|n} \in Z^+$

$\underline{|0} = \underline{|1} = 1$

${}^{n}P_0 = 1$

${}^{n}P_1 = n$

Example

4 Find the value of n in each of the following:

a ${}^{3n-5}P_5 = 2520$

b $\underline{|n+2} = 90\,\underline{|n}$

Solution

a $\because {}^{3n-5}P_5 = 2520 \quad \because {}^{7}P_5 = 2520$

$\therefore {}^{3n-5}P_5 = {}^{7}P_5 \quad \therefore 3n - 5 = 7$

$\therefore n = 4$

b $\because \underline{|n+2} = 90\,\underline{|n} \quad \therefore \dfrac{\underline{|n+2}}{\underline{|n}} = 90$

$\therefore {}^{n+2}P_2 = {}^{10}P_2 \quad \therefore n + 2 = 10$

$\therefore n = 8$

Try to solve

4 Find the value of r in each of the following: a ${}^{8}P_{3r-1} = 6720$ b ${}^{9}P_{r-4} = \underline{|9}$

Example

(5) If $^8P_{n-3} = {}^8P_{n-3}$, find the value of n.

Solution

$\because\ {}^8P_{n-3} = {}^8P_{n-3}$ $\quad \therefore\ 8 \leqslant n-3 \leqslant 0$ $\quad \therefore\ 3 \leqslant n \leqslant 11$ $\quad \therefore\ n \in \{3, 4, 5, 6, \dots, 10, 11\}$

Try to solve

(5) If $^{n-4}P_9 = {}^{n-4}P$, find the value of n

Example

(6) Find the value of n if $^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$

Solution

$\because\ {}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$ $\qquad \therefore\ \dfrac{\lfloor 56}{\lfloor 50 - r} \times \dfrac{\lfloor 51 - r}{\lfloor 54} = 30800$

$\therefore\ 56 \times 55\,(51 - r) = 30800$ $\qquad \therefore\ 51 - r = 10$ $\quad \therefore\ r = 41$

Try to solve

(6) If $^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$, find value of n.

Example

(7) If $^{n-2}P_{r-2} : {}^{n-2}P_r = 1 : 42$ and $^7P_{r-3} = 840$, find value of each of n, r?

Solution

$\because\ {}^7P_{r-3} = 840$, $\quad \because\ {}^7P_4 = 840$

$\therefore\ {}^7P_{r-3} = {}^7P_4$ $\quad \therefore\ r - 3 = 4$ $\quad \therefore\ r = 7$

$^{n-2}P_5 : {}^{n-2}P_7 = 1 : 42$ $\quad \therefore\ \dfrac{\lfloor n-2}{\lfloor n-7} \times \dfrac{\lfloor n-9}{\lfloor n-2} = \dfrac{1}{42}$

$(n - 7)(n - 8) = 42$ $\quad {}^{n-7}P_2 = {}^7P_2$ $\quad \therefore\ n - 7 = 7$ $\quad n = 14$

Try to solve

(7) Find the values of n, r in each of the following:

a $^{n-r}P_2 = 90$, $^{n+r}P_2 = 380$

b $^nP_r = 60480$, $\lfloor r = 720$

Example

(8) If $^nP_r = 120$ find all possible values of n, r

Solution

First: $^nP_r = 6 \times 5 \times 4 = {}^6P_3$ $\quad \therefore\ n = 6$ when $r = 3$

Second: $^nP_r = 5 \times 4 \times 3 \times 2 = {}^5P_4$ $\quad \therefore\ n = 5$ when $r = 4$

Third: $^nP_r = 5 \times 4 \times 3 \times 2 \times 1 = {}^5P_5$ $\quad \therefore\ n = 5$ when $r = 5$

Fourth: $^nP_r = {}^{120}P_1$ $\quad \therefore\ n = 120$ when $r = 1$

Try to solve

8) If ${}^{n}P_r = 210$, find all the possible values of n, r

Third: Combinations :

As you have studied the concept of combinations and known that combinations are all sets you can get from several objects by taking them all or taking some of them also you have studied the following relations :

1) ${}^{n}C_r = \dfrac{{}^{n}P_r}{\underline{|r}}$ for each $n \geqslant r$, $n \in N$, $r \in N$

2) ${}^{n}C_r = \dfrac{\underline{|n}}{\underline{|r}\ \underline{|n-r}}$ for each $n \geqslant r$, $n \in N$, $r \in N$

3) ${}^{n}C_r = {}^{n}C_{n-r}$ **(4)** If ${}^{n}C_x = {}^{n}C_y$ $\therefore x = y$ or $x + y = n$

Notice

${}^{n}C_n = {}^{n}C_0 = 1$

${}^{n}C_r \leqslant {}^{n}P_r$

${}^{n}C_r \in Z^+$

Example

9) Find the value of n in each of the following:

a) ${}^{n}C_{n-3} = 120$ b) ${}^{25}C_{3n-5} = {}^{25}C_{2n}$

Solution

a) $\because {}^{n}C_{n-3} = 120$ $\therefore {}^{n}C_3 = 120$ $\because 120 = {}^{10}C_3$

$\therefore {}^{n}C_3 = {}^{10}C_3$ $\therefore n = 10$

b) $\because {}^{25}C_{3n-5} = {}^{25}C_{2n}$

first: $3n - 5 = 2n$ $\therefore n = 5$ (check)

second: $3n - 5 + 2n = 25$ $5n = 30$ $\therefore n = 6$ check

Try to solve

9) Find the value of n in each of the following:

a) ${}^{n+1}C_{n-1} = 66$ b) ${}^{25}C_{2n-14} = {}^{25}C_{n-1}$

Ratio rule

$$\frac{{}^{n}C_r}{{}^{n}C_{r-1}} = \frac{n - r + 1}{r}$$

Notice

(1) $\dfrac{{}^{27}C_{14}}{{}^{27}C_{13}} = \dfrac{27 - 13}{14} = 1$

(2) $\dfrac{{}^{36}C_{25}}{{}^{36}C_{23}} = \dfrac{{}^{36}C_{25}}{{}^{36}C_{24}} \times \dfrac{{}^{36}C_{24}}{{}^{36}C_{23}}$

The previous relation can be proved as follows :

R.H.S : $\dfrac{{}^{n}C_r}{{}^{n}C_{r-1}} = \dfrac{\underline{|n}}{\underline{|r}\ \underline{|n-r}} \div \dfrac{\underline{|n}}{\underline{|r-1}\ \underline{|n-r+1}}$

$= \dfrac{\underline{|n}}{\underline{|r}\ \underline{|n-r}} \times \dfrac{\underline{|r-1}\ \underline{|n-r+1}}{\underline{|n}}$

$= \dfrac{1}{r\,\underline{|r-1}\ \underline{|n-r}} \times \dfrac{\underline{|r-1}\ (n-r+1)\,\underline{|n-r}}{1} = \dfrac{n-r+1}{r}$

Example

(10) If ${}^nC_6 : {}^nC_5 = \frac{1}{3}$, find the value of $\lfloor n - 2$

Solution

$\because \frac{{}^nC_6}{{}^nC_5} = \frac{1}{3}$ $\therefore \frac{n - 6 + 1}{6} = \frac{1}{3}$ $\therefore \frac{n - 5}{6} = \frac{1}{3}$ $\therefore n = 7$

$\therefore \lfloor n - 2 = \lfloor 7 - 2 = \lfloor 5 = 120$

Try to solve

(10) Calculate the value of r if : ${}^7C_r : {}^7C_{r-1} = \frac{1}{3}$

Addition rule

(2) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

The previous relation can be proved as follows:

$$\text{R.H.S} = \frac{\lfloor n}{\lfloor r \lfloor n - r} + \frac{\lfloor n}{\lfloor r - 1 \lfloor n - r + 1} = \frac{(n - r + 1)\lfloor n + r\lfloor n}{\lfloor r \lfloor n - r + 1} = \frac{(n - r + 1 + r)\lfloor n}{\lfloor r \lfloor n - r + 1}$$

$$= \frac{(n + 1)\lfloor n}{\lfloor r \lfloor n + 1 - r} = \frac{\lfloor n + 1}{\lfloor r \lfloor n + 1 - r} = {}^{n+1}Cr = \text{L.H.S.}$$

Example

(11) If ${}^{n+1}P_{r+1} : {}^{n+1}C_{r+1} = 720 : 1$, ${}^nC_{r-2} + {}^nC_{r-3} = 56$, find the numerical value of n and r.

Solution

${}^{n+1}P_{r+1} : {}^{n+1}C_{r+1} = 720 : 1$

$\lfloor r + 1 = 720$ $\lfloor r + 1 = \lfloor 6$ $\therefore r = 5$

$\because {}^nC_{r-2} + {}^nC_{r-3} = 56$ $\therefore {}^nC_3 + {}^nC_2 = 56$ $\therefore {}^{n+1}C_3 = 56$

$\therefore {}^{n+1}C_3 = {}^8C_3$ $\therefore n + 1 = 8$ $\therefore n = 7$

Try to solve

(11) If ${}^{13}C_r : {}^{13}C_{r+1} = 9 : 5$, ${}^nC_{r-2} + {}^nC_{r-1} = 3432$ **find** n , r

Example

(12) **Prove that** ${}^nC_r + 2\,{}^nC_{r+1} + {}^nC_{r+2} = {}^{n+2}C_{r+2}$ hence find the value of : ${}^{10}C_5 + 2\,{}^{10}C_6 + {}^{10}C_7$

Solution

$$\text{R.H.S} = \underbrace{{}^nC_r + {}^nC_{r+1}} + \underbrace{{}^nC_{r+1} + {}^nC_{r+2}}$$

$$= {}^{n+1}C_{r+1} + {}^{n+1}C_{r+2} = {}^{n+2}C_{r+2} = \text{L.H.S}$$

The expression $= {}^{10}C_5 + 2\,{}^{10}C_6 + {}^{10}C_7$ put $n = 10$, $r = 5$

From the previous relation, then : ${}^{10}C_5 + 2\,{}^{10}C_{5+1} + {}^{10}C_{5+2} = {}^{10+2}C_{5+2} = {}^{12}C_7 = 792$

Try to solve

12 a **Find the value of:** n if ${}^nC_3 + 2x\ {}^nC_2 + {}^nC_1 = 120$

b **Find the value of:** $\frac{{}^{17}C_6 + {}^{17}C_5}{{}^{18}C_5}$

Critical thinking: **Prove that** ${}^nC_r : {}^{n-1}C_{r-1} = \frac{n}{r}$ **and hence prove that** $\frac{{}^{25}C_4 + {}^{24}C_3}{{}^{24}C_3 + {}^{23}C_2} = \frac{58}{9}$

Example

13 If ${}^nC_{r+2} \times {}^nC_r \geqslant {}^nC_{r+1} \times {}^nC_{r-1}$, prove that : $n \geqslant 2r + 1$

Solution

$\because\ {}^nC_{r+2} \times {}^nC_r \geqslant {}^nC_{r+1} \times {}^nC_{r-1}$

$\therefore\ \frac{{}^nC_{r+2}}{{}^nC_{r+1}} \times \frac{{}^nC_r}{{}^nC_{r-1}} \geqslant 1 \Longrightarrow \frac{n-(r+1)+1}{r+2} \times \frac{n-(r+1)}{r} \geqslant 1$

$\frac{n-r-1}{r+2} \times \frac{n-r+1}{r} \geqslant 1 \Longrightarrow (n-r)^2 - 1 \geqslant r(r+2)$

$(n-r)^2 \geqslant (r+1)^2 \Longrightarrow n - r \geqslant r + 1 \qquad \therefore n \geqslant 2r + 1$

Try to solve

13 Find the possible values of n if ${}^nC_8 \times {}^nC_6 \geqslant {}^nC_7 \times {}^nC_5$

Exercises 1 - 1

First : Choose the correct answer from those given:

1 Number of ways to choose 2 different letters together or 3 different letters together from the elements of the set {a, b, c, d, e, h} is :

a ${}^6C_2 \times {}^6C_3$ b ${}^6P_2 \times {}^6P_3$ c ${}^6C_2 + {}^6C_3$ d ${}^6P_2 + {}^6P_3$

2 If ${}^nC_5 : {}^nC_4 = 3 : 1$, then n =

a 7 b 9 c 17 d 19

3 12 player participate in a swimming competition How many ways can the first, second and the third places be arranged?

a 1230 b 1320 c 2310 d 3210

4 Which of the following values could be equal to ${}^nP_2$?

a 24 b 25 c 27 d 30

5 If ${}^{17}P_{m+5} = {}^{29}P_{m+5}$, then m equals:

a -5 b 0 c 5 d 12

6 If ${}^{n-3}P_7 = \lfloor 8$, then n equals:

a 8 b 10 c 11 d 15

7) The value of $^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ equals:

a) $^{56}C_4$ b) $^{56}C_2$ c) $^{55}C_4$ d) $^{55}C_3$

8) The student should answer 10 questions out of 13 questions on a condition that he should answer 4 questions at least from the first 5 questions. How many ways can he answer?

a) 140   b) 196   c) 280   d) 346

9) If $^{n+2}C_4 = n^2 - 1$, then n equals:

a) 2   b) 4   c) 6   d) 10

Second : Answer the following questions:

10) How many ways can an even number and 2 odd numbers be selected from 4 even numbers and 5 odd numbers.

11) How many ways can an even number or 2 odd numbers be selected from 4 even numbers and 5 odd numbers.

12) How many ways can 8 awards be distributed on 4 students.

13) How many 4 - digit number can be formed from {1 , 2, 3, 4, 5, 6, 7}?

a) With replacement   b) Without replacement

14) If X = { 2, 3, 4, 5} and without repetition, find the number of numbers can be formed from X in each of the following.

a) If the number has 3-digit exactly.   b) If the number has 3-digit at least.

c) If the number has 3-digit at most.

15) Find the value of n, r in each of the following:

a) $^{n}P_2 = 2$, $^{n+r}P_2 = 90$ b) $^{n}P_4 = 840$, $^{8}P_r = 336$

c) $^{n}C_2 = 21$, $^{n+r}P_3 = 990$ d) $^{n}C_r = 10$, $\lfloor n-r = 6$

16) If $^{n}C_r : {}^{n}C_{r-1} = 2:3$, $^{n}C_{r-2} : {}^{n}C_{r-1} = 4:3$ find the numerical value of n, r.

17) Prove that $\frac{^{n}C_r}{^{n}C_r + {}^{n}C_{r+1}} = \frac{r+1}{n+1}$ and hence find value of $\frac{^{15}C_6 + {}^{15}C_7}{^{15}C_6}$.

18) Prove that $^{n}C_r : {}^{n-1}C_r = \frac{n}{n-r}$ and hence solve the equation $\frac{^{n}C_8 + {}^{n-1}C_8}{^{n-1}C_8} = 3$

19) If $^{n}C_r = {}^{n}C_{r-1}$, $^{n+1}C_r = \frac{40}{\lfloor r} \times {}^{n}P_{r-2}$, find n , r

20) If $^{n}P_r = 120\ {}^{n}C_{n-r}$ find value of $^{12}C_{2r}$, then find minimum value of n that makes the relation true.

21) If $^{n}C_{r-1} : {}^{n}C_r : {}^{n}C_{r+1} = 1:5:15$, find n, r.

22) Prove that $^{n+1}C_{r+1} \div {}^{n}C_r = \frac{n+1}{r+1}$, hence find the value of $\frac{^{15}C_5 + {}^{14}C_4}{^{14}C_4 + {}^{13}C_3}$

23 If $\frac{{}^{13}C_r + {}^{13}C_{r+1}}{{}^{13}C_{r-1} + {}^{13}C_r} = 2$, find the value of r

24 If ${}^{30}P_x = {}^{30}P_{x+2y}$, find x , y?

25 If ${}^{n}P_8 \geqslant {}^{n}P_7$, find n?

26 If: $|\underline{m+n} = 380\ |\underline{m+n-2}$, find m + n

27 Solve each of the following equations:

a $|\underline{n+2}\ \ |\underline{3n-4} = |\underline{n}\ \ |\underline{3n-2}$

b $2\ |\underline{2n} = (n^2 + 3n + 2)\ (|\underline{n})^2$

c $|\underline{2n}\ \ |\underline{n-3} = 72\ |\underline{3n-7}$

28 Prove that ${}^{n}C_r : {}^{n-1}C_{r-1} = \frac{n}{r}$ then find n, r if

${}^{n+2}C_{r+1} = 9\ {}^{n+1}C_r = 90 \times {}^{n}C_{r-1}$

29 **Sequence:**

a If $4 \times {}^{n}C_5$, $3 \times {}^{n}C_6$, $3 \times {}^{n}C_7$ form an arithmetic sequence, find n

b If $2 \times {}^{14}C_{r+1}$, $3 \times {}^{14}C_r$, $6 \times {}^{14}C_{r-1}$ form a geometric sequence, find r.

30 Find the values of n, r in each of the following:

a ${}^{n}C_r : {}^{n}C_{r+1} : {}^{n}C_{r+2} = 1 : 2 : 3$

b ${}^{n}C_r : {}^{n}C_{r+2} : {}^{n}C_{r+1} = 15 : 28 : 24$

c ${}^{n}C_r : {}^{n}C_{r+2} : {}^{n}C_{r+4} = 3 : 14 : 14$

d ${}^{n}C_r : {}^{n}C_{r-1} = 9 : 5$, ${}^{15}C_r = {}^{15}C_{3r-5}$

e ${}^{30}C_r = {}^{30}C_{r+10}$, ${}^{n}P_7 = 90 \times {}^{n-2}P_5$

31 4 noncolinear and coplaner points. Find the number of line segments joining each two of them?

32 How many ways can 3 persons be selected out of 5?

33 How many ways can a committee of 4 male students and 2 female students be selected out of 20 male students and 10 female students?

34 How many ways can a 7 member team be selected out of 9 girls and 5 boys if the members of the team has only 3 boys?

35 How many ways can 2 committees each of which formed from 3 persons be selected from 12 persons such that no one is chosen in the 2 committees?

36 Find the number of triangles formed from joining 3 vertices of a polygon whose number of sides is:

a 4 b 5 c 6

37 Find the number of diagonals of a polygon whose number of sides is:

a 6 b 8 c 12

38 A 4-person committee is to be formed out of 9 men and 3 women:

a Find the number of different ways to form this committe.

b How many committees that contain only one women?

c How many committees that contain at least one women?

Unit One

1 - 2

Binomial theorem for positive integer power

You will learn

- Relating pascal's triangle and coefficient of the expansion of the binomial
- The general form of the expansion of $(x + a)^n$ $n \in Z^+$
- The general form of the general term T_{+1} in the expansion of $(a + b)^n$
- The order and the value of the middle term and the two middle terms

Key terms

- The expansion
- Binomial
- The general term
- The middle term

Materials

- Scientific calculator
- Graphical programs

Think and discuss

We know that:

$(x + a)^1 = a + a$

$(x + a)^2 = x^2 + 2x\,a + a^2$

We can deduce that:

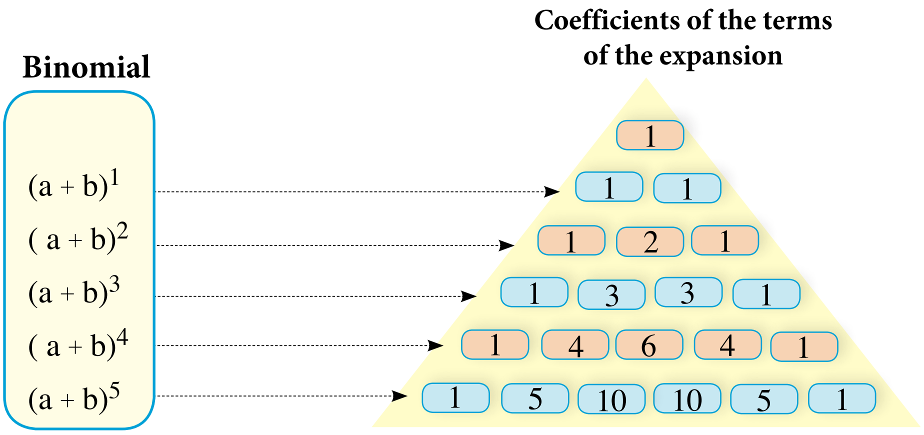
$(x + a)^3 = x^3 + 3x^2\,a + 3x\,a^2 + a^3$

$(x + a)^4 = x^4 + 4x^3\,a + 6x^2\,a^2 + 4x\,a^3 + a^4$

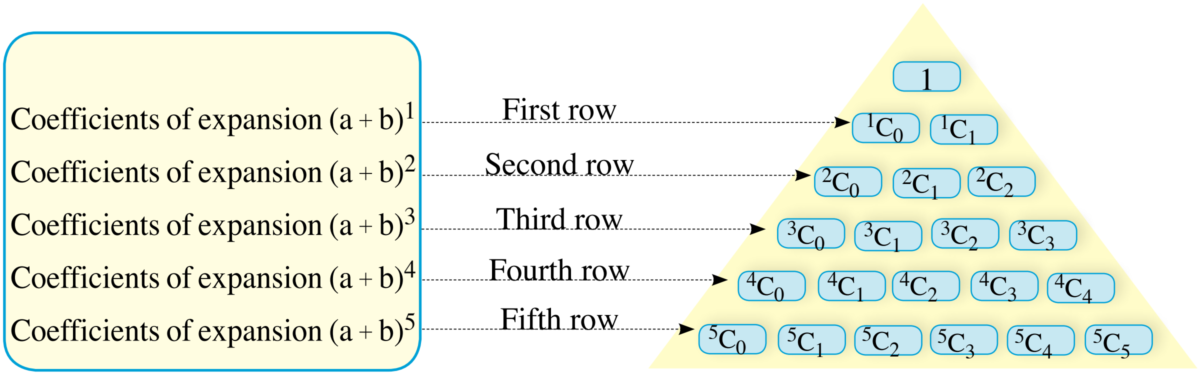
- What is the relation between the number of terms and the value of the power?
- What is the relation between the powers of the variables x, a in each term of the expansion ?
- What do you notice about the coefficients of terms in each term of the expansion?
- Can pascal's triangle be used to express the coefficients?
- Try to deduce a rule to expand $(a + b)^n$

Pascal's triangle

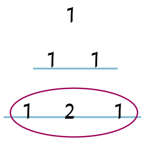
Notice that: **The coefficients of the expansion follow a pattern in pascal's triangle**



- Pascal's triangle can be written using the combinations as in the next figure:

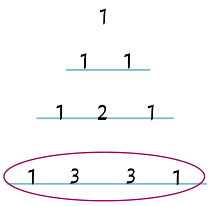


By noticing the second row (say) from bascal's triangle, we notice that 1, 2, 1 represent $^2C_0$, $^2C_1$, $^2C_2$ respectively and the sum of these elements $^2C_0$, $^2C_1$, $^2C_2$ represent the number of subsets which can be formed from a set containing two elements where $\quad {}^2C_0 + {}^2C_1 + {}^2C_2 = 4 = 2^2$



The set { x , y} all of its subsets are ϕ , {x} , {z},{x , z}

Similarly: the sum of elements of the third row $^3C_0$, $^3C_1$, $^3C_2$, $^3C_3$ represent the number of subsets which we obtained from a set contains three elements and the number of these sets is $8 = 2^3$



That is $\quad {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 = 2^3$

In general , if we have a set of n elements , then the number of subsets which can be obtained from its elements $= 2^n$ **i.e** $^nC_0 + {}^nC_1 + {}^nC_2 + \dots\dots + {}^nC_n = 2^n$

Verbal expression: using Bascal's triangle, find:

1) The coefficients of $(a+b)^6$ as combinations. **2)** The coefficients of $(a+b)^5$ as combinations.

Learn

The expansion of a binomial

If $a, x \in R$, $n \in Z^+$ then:

1- $(x+a)^n = x^n + {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 + \dots + a^n$

2- $(x-a)^n = x^n - {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 - \dots + (-a^n)$

Remarks on the expansion of the binomial $(x+a)^n$

1) The number of terms of the expansion is $(n+1)$ term.

2) The expansion is arranged in a descending order according to the powers of x and arranged in an ascending order according to the powers of a.

3) The sum of powers of x and powers of a in any term equals n.

4) The value of r in ^nC_r of each term is always decreased the order of the term by one.

Example
Writing the expansion of a binomial

(1) Write the expansion of $(2x + 3y)^4$

Solution

$(2x + 3y)^4 = (2x)^4 + {}^4C_1(2x)^3(3y) + {}^4C_2(2x)^2(3y)^2 + {}^4C_3(2x)(3y)^3 + (3y)^4$

$= 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$

Try to solve

(1) Write the expansion of:

a) $(3x + y)^5$ b) $(x^2 - 1)^6$

Special cases of the expansion of a binomial:

a) $(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + x^n$

b) $(1 - x)^n = 1 - {}^nC_1 x + {}^nC_2 x^2 - \ldots + (-x)^n$

Example

(2) Write the expansion of $(1 + x)^6$, then use it to find the numerical value of the expression:

$${}^6C_0 + {}^6C_1 + {}^6C_2 + \ldots\ldots + {}^6C_6$$

Solution

$(1 + x)^6 = 1 + {}^6C_1x + {}^6C_2x^2 + {}^6C_3x^3 + {}^6C_4x^4 + {}^6C_5x^5 + x^6$

Put x = 1 in both sides

$(1 + 1)^6 = 1 + {}^6C_1 + {}^6C_2 + {}^6C_3 + \ldots\ldots + 1$

$2^6 = {}^6C_0 + {}^6C_1 + {}^6C_2 + {}^6C_3 + \ldots\ldots + {}^6C_6$

Try to solve

(2) Write the expansion of $(1 - x)^8$, then use it to find the value of: $1 - {}^8C_1 + {}^8C_2 - {}^8C_3 + \ldots. + {}^8C_8$

Example

(3) Find the value of $(1{,}01)^9$, using the binomial theorem and approximate the result to three decimal numbers.

Solution

$(1{,}01)^9 = (1 + 0{,}01)^9$

$= 1 + {}^9C_1(\frac{1}{100}) + {}^9C_2(\frac{1}{100})^2 + {}^9C_3(\frac{1}{100})^3 + \ldots$

$= 1 + 0{,}09 + 0{,}0036 + 0{,}000084 + \ldots$ terms less than 0,001

$= 1.0936 \simeq 1.094$

Try to solve

(3) Find the value of $(0{,}98)^{10}$ by using the binomial theorem and approximate the result to three decimal numbers.

The general term of the expansion of a binomial

In the expansion of $(x + y)^n = x^n + {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}\, y^2 + ... + y^n$

Notice that $T_2 = {}^nC_1\, x^{n-1}\, y^1$, $T_3 = {}^nC_2 x^{n-2}\, y^2$

Similarly: $T_9 = {}^nC_8\, x^{n-8}\, y^8$

Let the general term be T_{r+1} where $0 \leqslant r \leqslant n$ then T_{r+1}. It can be written as:

$$T_{r+1} = {}^nC_r\, (x)^{n-r}\, (y)^r$$

Example

4 In the expansion of $(x + \frac{2}{x})^8$ **according to the descending power of x, find the coefficient of the sixth term**

Solution

$T_6 = {}^8C_5\, (x)^3\, (\frac{2}{x})^5 = {}^8C_5 \times 2^5\, x^{-2} = 1792\, x^{-2}$

The coefficient of this term $= 1792$

Notice the coefficient of $T_{r+1} = {}^nC_r$ (the coefficient of first term)$^{n-r}$ (coefficient of second term)^r

Try to solve

4 In the expansion of $(2x^2 + \frac{1}{2})^7$ according to the descending powers of x, find each of T_3 , T_7, and if $T_3 = T_7$, find the value of x.

Example

5 In the expansion of $(3x^2 - \frac{1}{2x})^{13}$ according to the descending power of x, find the tenth term from the end.

Solution

The tenth term from the end in the expansion of $(3x^2 - \frac{1}{2x})^{13}$ is the tenth term from the beginning of the expansion $(\frac{-1}{2x} + 3x^2)^{13}$

$$T_{10} = {}^{13}C_9 (\frac{-1}{2x})^4\, (3x^2)^9 = \frac{715 \times 3^9}{2^4}\, x^{14}$$

Another Solution

Notice that we can find the order of the tenth term from the end in the expansion $(3x^2 - \frac{-1}{2x})^{13}$, and its order equals $14 - 10 + 1 = 5$

T_{10} from the end is T_5 from the begining $T_5 = {}^{13}C_4 (3x^2)^9\, (\frac{-1}{2x})^4 = \frac{715 \times 3^9}{2^4}\, x^{14}$

Try to solve

5 In the expansion of $(2x - \frac{-1}{3x^2})^{11}$, find the fourth term from the end:

Rule

1) $(x + a)^n + (x - a)^n = 2\,(T_1 + T_3 + T_5 + ...)$ **2)** $(x + a)^n - (x - a)^n = 2\,(T_2 + T_4 + T_6 + ...)$

Example

6 Find in the simplest form $(x + 2)^6 + (x - 2)^6$

Solution

$(x + 2)^6 + (x - 2)^6 = 2\,(T_1 + T_3 + T_5 + T_7)$

$= 2\,(x^6 + {}^6C_2\, x^4 \times 2^2 + {}^6C_4\, x^2 \times 2^4 + 2^6) = 2\,(x^6 + 60x^4 + 240\,x^2 + 64)$

Try to solve

6 **1)** Find in the simplest form $(1 + \sqrt{x}\,)^5 - (1 - \sqrt{x}\,)^5$

2) Find to the nearest three decimals $(1{,}03)^8 + (0{,}97)^8$, using the binomial theorem.

Example

7 In the expansion $(3 + x)^{11} - {}^{11}C_1\,(3 + x)^{10}\,(1 - 2x) + {}^{11}C_2\,(3 + x)^9\,(1 - 2x)^2 - \ldots\ldots - (1 - 2x)^{11}$ find the fifth term.

Solution

The expression represents the expansion of $[(3 + x) - (1 - 2x)]^{11} = (2 + 3x)^{11}$

then:

$T_5 = {}^{11}C_4(2)^7\,(3x)^4 = 330 \times 2^7 \times 3^4\,x^4 = 3421440\,x^4$

Try to solve

7 In the expansion $(1 - x)^8 + 24\,x\,(1 - x)^7 + 252\,x^2\,(1 - x)^6 + \ldots\ldots + 6561x^8$, find the numerical value of the sixth term when $x = 1$

Example

8 If $(1 + cx)^n = 1 + 20\,x + a_1x^2 + a_2x^3 + \ldots + a_{n-1}\,x^n$

and $16\,a_1 = 3\,a_2$, find the value of n, c where c 0 $\neq$

Solution

$(1 + c\,x)^n = 1 + {}^nC_1\,cx + {}^nC_2\,c^2\,x^2 + {}^nC_3\,c^3\,x^3 + \ldots\ldots$

$\therefore {}^nC_1\,c = 20 \qquad \therefore n\,c = 20 \qquad \therefore c = \frac{20}{n}$ (1)

$\because 16 \times {}^nC_2\,c^2 = 3 \times {}^nC_3\,c^3 \qquad \therefore 16 \times {}^nC_2 = 3 \times {}^nC_3\,c$ (2)

Substituting from (1) in (2) $\qquad \therefore 16 \times {}^nC_2 = 3 \times {}^nC_3 \times \frac{20}{n}$

$\therefore 16\,n = 3 \times \frac{{}^nC_3}{{}^nC_2} \times 20 \qquad \therefore 16n = 3 \times \frac{n-2}{3} \times 20 \qquad \therefore n = 10$

Substituting equation (1) $\qquad \therefore c = \frac{20}{10} = 2$

Try to solve

8 In the expansion of $(1 + Kx)^{10}$, if the coefficient of the third term equals 180, and the coefficient of the fifth term equals 210, find the values of K and x where K is a positive integer.

Example

9 Find the coefficient of x^{10} in the expansion of $(1 + x - x^2)^9$

Solution

In the expansion of $(1 + (x - x^2))^9$

$\because\ T_{r+1} = {}^9C_r \times (x - x^2)^r$

$\therefore\ T_{r+1} = {}^9C_r \times x^r \times (1 - x)^r$

$\therefore\ T_{r+1} = {}^9C_r \times x^r \times {}^rC_m(-x)^m$

$\therefore\ T_{r+1} = (-1)^m \times {}^9C_r \times {}^rC_m \times x^{r+m}$

to find the coefficient of x^{10}

$r + m = 10$ where $m \leqslant r < 10$

r = 5	r = 6	r = 7	r = 8	r = 9
m = 5	m = 4	m = 3	m = 2	m = 1

Coefficient of $x^{10} = -{}^9C_5 \times {}^5C_5 + {}^9C_6 \times {}^6C_4 - {}^9C_7 \times {}^7C_3 + {}^9C_8 \times {}^8C_2 - {}^9C_9 \times {}^9C_1$

$\therefore$ Coefficient of $x^{10} = -126 + 1260 - 1260 + 252 - 9 = 117$

Try to solve

9 Find the coefficient of x^2 in the expansion $(1 + x + x^2)^5$

Example

10 **a** Prove that $= \dfrac{{}^nC_r}{{}^{n-1}C_{r-1}} = \dfrac{n}{r}$

b If the ratio between T_6 in the expansion of $(x + \frac{1}{x})15$ and T_5 in the expansion of $(x- \frac{1}{x^2})^{14}$ equals $\frac{8}{9}$, find the value of x

Solution

$${}^nC_r \div {}^{n-1}C_{r-1} = \frac{|\underline{n}}{|\underline{r}\ |\underline{n-r}} \times \frac{|\underline{r-1}\ |\underline{n-1-r+1}}{|\underline{n-1}} = \frac{n\,|\underline{n-1}\ |\underline{r-1}}{r\,|\underline{r-1}\ |\underline{n-1}} = \frac{n}{r}$$

$$= \frac{T_6 \text{ of } (x + \frac{1}{x})^{15}}{T_5 \text{ of } (x - \frac{1}{x^2})^{14}} = \frac{{}^{15}C_5\, x^{10} (\frac{1}{x})^5}{{}^{14}C_4\, x^{10} (\frac{-1}{x^2})^4} = \frac{8}{9}$$

$\therefore\ 3x^3 = \frac{8}{9}$ $\quad\therefore\ x^3 = \frac{8}{27}$ $\quad\therefore\ x = \frac{2}{3}$

Try to solve

(10) Use the binomial theorem to prove that $({}^{n}C_0)^2 + ({}^{n}C_1)^2 + ({}^{n}C_2)^2 + ... + ({}^{n}C_n)^2 = {}^{2n}C_n$

The middle term in the expansion of $(x + a)^n$

In the expansion of $(x + a)^n$,we find that the number of terms of the expansion = $n + 1$

First: If n is an even number , then the number of terms of the expansion is an odd number and the expansion has one middle term of order $\frac{n+2}{2}$ $\left(\frac{n}{2} + 1\right)$

Second: If n is odd number , then the number of terms of the expansion is an even number, and the expansion has two middle terms of order $\frac{n+1}{2}$, $\frac{n+3}{2}$ $\left(\frac{n+1}{2}\right.$, next term)

Example

(11) Find the middle term in the expansion of $\left(2x + \frac{1}{2x^2}\right)^{12}$

Solution

The order of middle term = $\frac{12}{2} + 1 = 7$

$$T_7 = {}^{12}C_6 (2x)^6 \left(\frac{1}{2x^2}\right)^6 = {}^{12}C_6 (2)^6 \left(\frac{1}{2}\right)^6 x^{6-12} = {}^{12}C_6\, x^{-6}$$

Try to solve

(11) Find the middle term in the expansion of $\left(x^2 + \frac{1}{2x}\right)^{10}$. If the value of this term = $\frac{28}{27}$, find the value of x

Example

(12) Find the two middle terms of the expansion of $\left(\frac{x^2}{3} + \frac{3}{x}\right)^{15}$

Solution

The orders of the two middle terms are $\frac{15+1}{2}$ and the next term i.e T_8 , T_9

$$T_8 = {}^{15}C_7 \left(\frac{x^2}{3}\right)^8 \left(\frac{3}{x}\right)^7 = {}^{15}C_7 \times 3^{-8+7} \times x^{16-7} = {}^{15}C_7 \times \frac{1}{3} x^9$$

$$T_9 = {}^{15}C_8 \left(\frac{x^2}{3}\right)^7 \left(\frac{3}{x}\right)^8 = {}^{15}C_8 \times {}^{8}3^{-7} \times s^{14-8} = {}^{15}C_8 \times 3x^6$$

Try to solve

(12) If the two middle terms of the expansion of $(3x + 2y)^{13}$ are equal, prove that $\frac{x}{y} = \frac{2}{3}$

Example

13 Find the middle term in the expansion of $(3 + 2y)^8 + (3 - 2x)^8$

Solution

The order of middle term $= \frac{8}{2} + 1 = 5$

$$T_5 = {}^8C_4\,(3)^4\,(2x)^4 + {}^8C_4(3)^4\,(-2x)^4$$

$$= 2 \times {}^8C_4\,(3)^4\,(2x)^4 = 181440\,x^4$$

Try to solve

13 Find the middle term in the expansion of $\left(2\sqrt{x} + \frac{1}{2\sqrt{x}}\right)^{10} + \left(2\sqrt{x} - \frac{1}{2\sqrt{x}}\right)^{10}$

Exercises 1 - 2

Choose the correct answer:

1 If the orders of the two middle terms of the expansion of $(x + y)^n$ are 7 and 8, then n equals:

a) 14 b) 15 c) 16 d) 56

2 If $1 + 5x + \frac{5 \times 4}{2 \times 1}x^2 + \frac{5 \times 4 \times 3}{3 \times 2 \times 1}x^3 + \dots + x^5 = 1024$, then the value of x equals:

a) 1 b) 2 c) 10 d) 3

3 Sum of the coefficients of terms of the expansion of $(x^2 - \frac{1}{x})^7$ equals:

a) 2^7 b) 2^5 c) 2^6 d) zero

4 The coefficient of the fifth term in the expansion of $(1 + 2x)^{10}$ is

a) $16\ {}^{10}C_5$ b) $\frac{1}{16}\ {}^{10}C_5$ c) $16\ {}^{10}C_4$ d) $\frac{1}{16}\ {}^{10}C_4$

5 In the expansion of a binomial theorem, if the general term is ${}^{12}C_r\ x^{24-4r}$, then the term containing x^{12} is:

a) T_3 b) T_4 c) T_5 d) Does not exist

6 If the two middle terms of the expansion of $(a + 2b)^{2n+1}$ are equal , then:

a) $\frac{a}{b} = \frac{1}{2}$ b) $a = 4b$ c) $a = 8b$ d) $a = 2b$

7 If the middle term of the expansion of $(\frac{2a}{3} + \frac{b}{a^2})^{8n}$ is the ninth term , then n equals:

a) 1 b) 2 c) 3 d) 4

8. In the expansion of $(1 + bx)^9$, the coefficient of the sixth term is:
   a) ${}^9C_5$ b) ${}^9C_6$ c) ${}^9C_5\, b^5$ d) ${}^9C_6\, b^6$

9. In the expansion of a binomial , we have 7 positive terms and 6 negative terms , then the term is in the form of:
   a) $(a - b)^{12}$ b) $(a - b)^{13}$ c) $(a + b)^{12}$ d) $(a - b)^{13}$

Second : Answer the following:

10. If $1 + 8x + {}^8C_2\, x^2 + \dots\dots + x^8 = 256$, find the value of x

11. Find to the nearest thousand using the binomial theorem the value for each of:
   a) $(1.003)^5$ b) $(0.998)^7$ c) $(1.01)^6 + (00.99)^6$ d) $(1.02)^8 - (0.98)^8$

12. Find the value of x which satisfies $(1 + \sqrt{3})^6 - (1 - \sqrt{3})^6 = 480\sqrt{3}\,x$

13. Use the expansion of : $(1 + x)^{10} = 1 + {}^{10}C_1\, x + {}^{10}C_2\, x^2 + \dots\dots + x^{10}$ to prove that:
   a) $1 + {}^{10}C_1 + {}^{10}C_2 + \dots\dots + {}^{10}C_{10} = 2^{10}$ b) $1 - {}^{10}C_1 + {}^{10}C_2 - \dots\dots + {}^{10}C_{10} = 0$

14. Write the expansion for each of :
   a) $(\frac{2}{x} + \frac{x}{2})^4$ b) $(x - \frac{1}{x})5$
   c) $(x + \sqrt{2})^4 + (x - \sqrt{2})^4$ d) $(\sqrt{3} + 2x)^5 - (\sqrt{3} - 2x)^5$

15. In the expansion of $(1 + x)^n$ according to the ascending powers of x, if $T_3 = 28x^2$, $T_5 = 1120$, find the value of n and x

16. In the expansion of $(1 + x)^n$, if the coefficient of the sixth term equals the coefficient of the tenth term, find the value of n.

17. In the expansion of $(a\,x + b)^{10}$ according to the descending powers of x, if the coefficient of $T_6 = \frac{63}{8}$, prove that $2\,ab = 1$

18. In the expansion of $(2x^2 + \frac{1}{2x})^{12}$, find the middle term.

19. In the expansion of $(\frac{x^2}{2} - \frac{2}{x})^{11}$,find the two middle terms.

20. In the expansion of $x^4 (x - \frac{1}{x})^9$ according to the descending powers of x, find the fourth term from the end.

21. If the middle term of the expansion of $(x^2 + \frac{1}{2x})^{10}$ equals $\frac{28}{27}$, find the value of x.

22. Find the ratio between the middle term and the fifth term in the expansion of $(\frac{2x}{3} + \frac{3}{2x})^{10}$, then find the numerical value of the ratio when $x = 3$

23. If the ratio between the fifth term in the expansion of $(x + \frac{1}{x})^{15}$ and the fourth term in the expansion of $(x - \frac{1}{x^2})^{14}$ equals -16 : 15, find the value of x.

Unit One

1 - 3 Finding the term containing x^R in the expansion of binomial

You will learn

- *Using the general term to find the term containing x^k and term free of x.*
- *Finding the coefficient of the term containing x^k.*
- *Finding the highest coefficient of the expansion.*

Key terms

- *General term*
- *Term free of x*
- *Heightest power*
- *Coefficient*

Materials

- *Scientific calculator*

Think and discuss

We studied in the previous lesson that :

$$(x^2 - \frac{1}{2x})^{20} = (x^2)^{20} - {}^{20}C_1(x^2)^{19}(\frac{1}{2x}) + {}^{20}C_2(x^2)^{18}(\frac{1}{2x})^2 - {}^{20}C_3(x^2)^{17}(\frac{1}{2x})^3 + \dots\dots + (\frac{-1}{2x})^{20}$$

Is it easy to find the term containing x^{16} or x^{24} or term free of x without writing the terms of the expansion?

We notice that it is a difficult method to find the term containing x^k by writing all the terms of the expansion , so we follow the next :

1- Suppose that this term is the general term T_{r+1} , we find this term in terms of r.

2- Find the sum of the powers of x in the general term in terms of r and place this sum to be equal to the required power k then find r which satisfies the inclusion of this term on the required power k and we have::

a If $r \in N$, then $r + 1$ is the required term.

b If $r \notin N$ then there is no term containing the required power in this expansion.

To find the term free of x, we put the sum of powers of x in the general term = 0

Example

1 In the expansion of $(\frac{3x}{2} + \frac{2}{3x})^{11}$, find the coefficient of x in this expansion.

Solution

$$T_{r+1} = {}^{11}C_r\,(\frac{3x}{2})^{11-r}\,(\frac{2}{3x})^r$$

Compairing the powers $x^{11-r-r} = x^1$

$11 - 2r = 1 \qquad r = 5$

Required term is the sixth term coefficient of $T_6 = {}^{11}C_5\,(\frac{3}{2})^6\,(\frac{2}{3})^5 = 693$

Try to solve

1. Find the coefficient of x^8 in the expansion of $(\frac{2x}{3} - \frac{3}{x})^{12}$

Example

2. In the expansion of $(2x - \frac{1}{2x^2})^9$, find :

a. The coefficient of x^3

b. The term free of x

c. Prove that the term does not contain x^2

Solution

$T_{r+1} = {}^9C_r\,(2x)^{9-r}\left(\frac{-1}{2x^2}\right)^r = {}^9C_r\,(-1)^r\,(2)^{9-2r}\,x^{9-3r}$

Comparing the powers

a. To find the coefficient of x^3

$x^{9-3r} = x^3 \qquad 9 - 3r = 3 \qquad r = 2$

the third term contains x^3

the coefficient of $T_3 = {}^9C_2 \times 2^5 = 36 \times 32 = 1152$

b. To find the term free of x $\qquad 9 - 3r = 0 \qquad r = 3$

Required term is $T_4 = {}^9C_3\,(-1)^3\,(2)^3 = -672$

c. Put $9 - 3r = 2 \qquad \therefore 7 = 3r \qquad \therefore r = \frac{7}{3} \notin N$

$\therefore$ This expansion does not contain x^2

Try to solve

2. a. Find the term free of x in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$

b. Find the coefficient of x^{-10} in the expansion of $\left(\frac{x^2}{3} - \frac{2}{x^2}\right)^{15}$

c. In the expansion of $\left(ax + \frac{1}{bx}\right)^{10}$ according to the descending power of x , if the term free of x equals the coefficient of the seventh term, Prove that $6ab = 5$

Example

3. If n is a positive integer, prove that there is no term free of x in the expansion of $\left(x^5 + \frac{1}{x^2}\right)^n$ except if n is a multiple of 7 , then find this term in the case of n = 7

Solution

$T_{r+1} = {}^nC_r\,(x^5)^{n-r}\left(\frac{1}{x^2}\right)^r = {}^nC_r\; x^{5n-7r}$

$x^{5n-7r} = x^0 \qquad 5n - 7r = 0 \qquad r = \frac{5n}{7}$

$\frac{5n}{7} \in Z^+$ when n is a multiple of 7 $\qquad$ when $n = 7$

$r = 5 \qquad$ the term free of x is T_6

$T_6 = {}^7C_5 = 21$

Try to solve

3) In the expansion of $\left(x^2 + \frac{1}{x}\right)^{3n}$ find :

a) The coefficient of the term which contains x^{3n}

b) If $n = 6$, find the ratio between the coefficient of the term containing x^{3n} and the coefficient of the middle term

Example

4) In the expansion of $\left(2 + \frac{x}{3}\right)^9$, find the value of x which makes the two middle terms equal.

Solution

The order of the two middle terms $\frac{9+1}{2}$ and the consecutive i.e. T_5 , T_6

$\because\ T_5 = T_6 \qquad \therefore\ {}^9C_4\,(2)^5\left(\frac{x}{3}\right)^4 = {}^9C_5(2)^4\left(\frac{x}{3}\right)^5$

$2 = \frac{x}{3} \qquad \therefore\ x = 6$

Try to solve

4) Find the coefficient of the middle term in the expansion of $(1 + 3x + 3x^2 + x^3)^4$

Exercises 1 - 3

Choose the correct answer:

1) The term containing x^4 in the expansion of $(1 + 2x)^{10}$ equals:

a) ${}^{15}C_4$ $\qquad$ b) $\frac{1}{16}\,{}^{10}C_4$ $\qquad$ c) $16\,{}^{10}C_4$ $\qquad$ d) $32\,{}^{10}C_5$

2) In the expansion of $\left(x + \frac{1}{x}\right)^{10}$, the term free of x is:

a) T_4 $\qquad$ b) T_5

c) T_6 $\qquad$ d) not found

3. In the expansion of $x^3 (1 + x)^7$, the coefficient of the term containing x^4 is:

a. ${}^7C^4$ b. ${}^7C_3$ c. ${}^7C_1$ d. 21

4. In the expansion of $(x^2 + \frac{2}{x})^6$, the term free of x is

a. Third. b. Fourth.
c. Fifth. d. not found

5. In the expansion of $(ax^2 + \frac{1}{ax})^{11}$, if the coefficients of x^4 and x^7 are equal , then a =

a. 1 b. -1 c. ±1 d. ±2

6. If the term free of x in the expansion of $(x + \frac{1}{x})^n$ is T_7 , then n =

a. 6 b. 10 c. 12 d. 8

7. In the expansion of $(x^2 + \frac{1}{ax})^8$, if the coefficient of the middle term equals the coefficient of x^7 , then a =

a. $\frac{4}{5}$ b. $\frac{-4}{5}$ c. $\frac{-5}{4}$ d. $\frac{5}{4}$

8. In the expansion of $(ax + \frac{1}{bx})^{10}$ according to the descending powers of x, if the term free of x equals the coefficient of the seventh term, then:

a. $ab = \frac{6}{5}$ b. $ab = \frac{5}{6}$ c. $ab = \frac{36}{25}$ d. $ab = \frac{25}{36}$

9. The term free of x in the expansion of $(2x + \frac{1}{2x})^8$

a. 35 b. 140 c. 70 d. 56

10. In the expansion of $(1 + ax)^7$ according to the ascending powers of x, if the coefficient of $T_5 = 560$, then a =

a. 2 b. 4 c. ±2 d. ±4

Answer the following questions :

11. In the expansion of $(4x^2 + \frac{1}{2x})^{12}$, find the term free of x

12. Find the coefficient of x^{12} in the expansion of $x^2 (\frac{x^2}{2} + \frac{2}{x^3})^{15}$

13. If the sixth term of the expansion $(2x - \frac{1}{x^3})^n$ according to the descending powers of x is free of x, find the value of n. Then investigate if any of these terms of this expansion contains x^{-6} or not.

14. In the expansion of $(2x - \frac{1}{x^2})^9$:

first: find the coefficient of x^3 **second :** find the term free of x

third: Prove that this expansion does not contain a term containing x^2

15. Prove that ${}^{n}C_r : {}^{n-1}C_{r-1} = \frac{n}{r}$ and if the ratio between the coefficient of T_{11} in the expansion of $(1 + x^2)^n$ and the coefficient of T_{10} in the expansion of $(1 - y)^{n-1}$ equals -3 : 2, find the value of n.

16. Find the coefficient of $(\frac{x}{y})^4$ in the expansion of $(\frac{2x}{y} + \frac{y}{2x})^{10}$

17. Find the coefficient of x^n in the expansion of $(1 + x)^{2n}$, then prove that it is equal to twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$

18. In the expansion of $(x + \frac{1}{x})^{2n}$, prove that the term free of x is the middle term, then find the value of this term when $n = 8$

19. In the expansion of $(x^k + \frac{1}{x})^6$ where k is a positive integer , find :

    First: The value of k which makes the expansion have a term free of x

    Second: The ratio between the term free of x and the coefficient of the middle term at the highest value of k you have got from First.

20. In the expansion of $(x^2 + \frac{1}{ax})^{12}$, if the ratio between the term free of x and the coefficient of x^3 in this expansion equals 5 : 16 , find the value of a , then find the value of the middle term when $x = 2$.

21. In the expansion of $(2x^2 + \frac{a}{x^3})^{10}$, if the coefficient of x^5 equals the coefficient of x^{15} , find the value of a.

22. In the expansion of $(x^2 + \frac{1}{8x})^{13}$ according to the descending power of x :

    First: Prove that there is no term free of x **Second:** If $T_4 = T_{11}$, find the value of x

23. In the expansion of $(x + \frac{1}{x^2})^9$ find:

    First: The order and the value of the term free of x

    Second: The value of x which makes the sum of the two middle terms in the expansion equal to zero.

24. Find the value of the term free of x in the expansion of $(9x^2 + \frac{1}{3x})^9$, then find the value of x which makes the two middle terms equal.

25. In the expansion of $(x^2 + \frac{1}{x})^{3n}$, prove that the term free of x equals the coefficient of the term containing x^{3n} , if n = 6, find the ratio between the term free of x and the coefficient of the middle term.

Ratio between two consecutive terms in the binomial expansion

Unit One

1 - 4

You will learn

- Find the ratio between two consecutive terms.
- Find the ratio between the two coefficients of two consecutive terms

Key terms

- Consecutive terms

Materials

- Scientific calculator

In the expansion of $(x + a)^n$ and the two consecutive terms are T_{r+1} , Tr

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r\,(X)^{n-r}\,(a)^r}{{}^nC_{r-1}\,(X)^{n-r+1}(a)^{r-1}}$$

$$= \frac{{}^nC_r}{{}^nC_{r-1}} \times \frac{a}{x}$$

$$\boxed{\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \times \frac{a}{x}}$$

and : $\dfrac{\text{coefficient of } T_{r+1}}{\text{coefficient of } T_r} = \dfrac{n-r+1}{r} \times \dfrac{\text{coefficient of a}}{\text{coefficient of x}}$

Example

1 In the expansion of $(3 + 2y)^{12}$ find each of :

a $\dfrac{T_3}{T_2}$ b $\dfrac{\text{coefficient of } T_7}{\text{coefficient of } T_8}$

c $\dfrac{T_6}{T_4}$ d $\dfrac{\text{coefficient of } T_8}{\text{coefficient of } T_6}$

Solution

a $\dfrac{T_3}{T_2} = \dfrac{12-2+1}{2} \times (\dfrac{2y}{x})$

$= \dfrac{11}{2} \times \dfrac{2y}{x} = \dfrac{11y}{x}$

b $\dfrac{\text{coefficient of } T_7}{\text{coefficient of } T_8} = \dfrac{7}{12-7+1} \times \dfrac{1}{2} = \dfrac{7}{12}$

c $\dfrac{T_6}{T_4} = \dfrac{T_6}{T_5} \times \dfrac{T_5}{T_4}$

$= \dfrac{12-5+1}{5} \times (\dfrac{2y}{x}) \times \dfrac{12-4+1}{4} (\dfrac{2y}{x})$

$= \dfrac{8}{5} \times \dfrac{2y}{x} \times \dfrac{9}{4} \times \dfrac{2y}{x} = \dfrac{72\,y^2}{5\,x^2}$

d $\dfrac{\text{coefficient of } T_8}{\text{coefficient of } T_6} = \dfrac{\text{coefficient of } T_8}{\text{coefficient of } T_7} \times \dfrac{\text{coefficient of } T_7}{\text{coefficient of } T_6}$

$= \dfrac{12-7+1}{7} \times \dfrac{2}{1} \times \dfrac{12-6+1}{6} \times \dfrac{2}{1}$

$= \dfrac{6}{7} \times \dfrac{2}{1} \times \dfrac{7}{6} \times \dfrac{2}{1} = 4$

Try to solve

1. In the expansion of $(x^2 + \frac{2}{x})^8$

First: Find the ratio between the fifth and the sixth terms. If this ratio equals 8 : 25, find the value of x

Second: Prove that this expansion does not have a term free of x

Example

2. In the expansion of $(x + y)^8$ if $2T_5 = T_4 + T_6$, find $\frac{x}{y}$ numerically.

Solution

$T_4 + T_6 = 2T_5$ by dividing by T_5 $\qquad \frac{T_4}{T_5} + \frac{T_6}{T_5} = 2$

$\frac{4}{8-4+1}(\frac{x}{y}) + \frac{8-5+1}{5}(\frac{x}{y}) = 2 \qquad \frac{4x}{5y} + \frac{4y}{5x} = \frac{2}{1}$ **by multiplying by** $5xy$

$4x^2 + 4y^2 = 10xy \qquad 4x^2 - 10xy + 4y^2 = 0 \div 2$

$2x^2 - 5xy + 2y^2 = 0 \qquad (2x - y)(x - 2y) = 0$

$2x = y \qquad x = 2y$

$\frac{x}{y} = \frac{1}{2}$ أو $\frac{x}{y} = \frac{2}{1}$

Try to solve

2. In the expansion of $(\sqrt{x} + \frac{1}{x})^8$, If $T_4, T_5, 25T_7$ are proportional, find the value of x.

Example

3. If the coefficients of three consecutive terms of the expansion of $(1 + x)^n$ are 35, 21, 7 according to the ascending power of x, find the value for each of n and the orders of these three terms.

Solution

Let the terms T_r, T_{r+1}, T_{r+2}

$\frac{\text{coefficient of } T_{r+1}}{\text{coefficient of } T_r} = \frac{n-r+1}{r} = \frac{21}{35} \qquad \frac{n-r+1}{r} = \frac{3}{5}$

$5n - 5r + 5 = 3r \qquad \boxed{5n - 8r = -5} \qquad (1)$

$\frac{\text{coefficient of } T_{r+2}}{\text{coefficient of } T_{r+1}} = \frac{n-(r+1)+1}{r+1} = \frac{7}{21} \qquad \frac{n-r}{r+1} = \frac{1}{3}$

$3n - 3r = r + 1 \qquad \boxed{3n - 4r = 1} \qquad (2)$

By solving the two equations: (1) and (2) $\quad \therefore n = 7, r = 5$

Try to solve

3. If the third, fourth and fifth terms of the expansion $(x + y)^n$ are 112, 448, 1120 respectively, find the values of each of n, y, x

Example **Finding the greatest term**

4 Find the coefficient of the greatest term in the expansion of $(2x + 3y)^{10}$

Solution

$\because \dfrac{T_{r+1}}{T_r} = \dfrac{10 + 1 - r}{r} \times \dfrac{3}{2}$ $\therefore \dfrac{T_{r+1}}{T_r} = \dfrac{11 - r}{r} \times \dfrac{3}{2} = \dfrac{33 - 3r}{2r}$

First: $\dfrac{33 - 3r}{2r} \geqslant 1$ $\therefore 33 - 3r \geqslant 2r$ $\therefore 5r \leqslant 33$ $\therefore r \leqslant 6.6$

From this, we deduce that $T_7 > T_6 > T_5 > ..... > T_1$

Second: $\dfrac{33 - 3r}{2r} \leqslant 1$ $\therefore 33 - 3r \leqslant 2r$ $\therefore 5r \geqslant 33$ $\therefore r \geqslant 6.6$

From this, we deduce that $T_{11} < T_{10} < T_9 < T_8 < T_7$

$\therefore T_7$ is the greatest term in the expansion of $(2x + 3y)^{10}$

$\therefore$ coefficient of $T_7 = {}^{10}C_6 \times 2^4 \times 3^6$ $\therefore T_7 = 2449440$

Exercises 1 - 4

Choose the correct answer from the given answers:

1. In the expansion of $(x + y)^{10}$, the ninth term : the eighth term equals ....
   - a $\dfrac{3y}{8x}$
   - b $\dfrac{3x}{8y}$
   - c $\dfrac{8y}{3x}$
   - d $\dfrac{8x}{3y}$

2. In the expansion of $(1 - x)^{12}$ the coefficient of sixth term : the coefficient of fifth term ....
   - a $\dfrac{8}{5}$
   - b $\dfrac{5}{8}$
   - c $\dfrac{-8}{5}$
   - d $\dfrac{-5}{8}$

3. In the expansion of $(x + y)^8$, then the ratio $\dfrac{T_6}{T_4}$ = ...
   - a $\dfrac{25y^2}{16x^2}$
   - b $\dfrac{25x^2}{16y^2}$
   - c 1
   - d $\dfrac{y^2}{x^2}$

4. In the expansion of $(3a - 2b)^{11}$, if the ratio between the two middle terms respectively equals $\dfrac{-3}{2}$, then a : b = ...
   - a 9 : 4
   - b 4 : 9
   - c 1
   - d -1

Second : Answer the following questions:

5. In the expansion of $\left(2x^2 + \dfrac{3}{x^2}\right)^{11}$, find each of:
   - a $\dfrac{T_3}{T_2}$
   - b $\dfrac{T_4}{T_5}$
   - c $\dfrac{T_6}{T_8}$
   - d $\dfrac{\text{coefficient of } T_4}{\text{coefficient of } T_6}$

6. In the expansion of $(1 + x)^{12}$, if $T_3 = 2T_2$, find the value of x ....

7. In the expansion of $(a + b)^n$ if $T_2 = 240$, $T_3 = 720$, $T_4 = 1080$, find the value of each of a, b, n

8. If $T_2 : T_3$ in the expansion of $(a + b)^n$ equals the ratio $T_3 : T_4$ in the expansion of $(a + b)^{n+3}$, find the value of n

9. In the expansion of $(1 + mx)^n$, if $4T_6 = 7T_8$, $\frac{T_4}{T_6} = \frac{1}{4}$ when $x = 1$, find the value of m, n
10. Find numerically the value of the greatest term in the expansion of $(3 - 5x)^{15}$ when $x = \frac{1}{5}$
11. In the expansion of $(x + y)^n$ according to the descending power of x, if the second term is the arithmetic mean between the first and the third terms when $x = 2y$, find the value of n.

Unit Summary

1. $^nP_r = n(n-1)(n-2) \ldots (n-r+1)$
2. $^nP_r = \frac{\lfloor n}{\lfloor n-r}$
3. $\lfloor 1 = \lfloor 0 = 1$
4. $^nC_r = \frac{^nP_r}{\lfloor r} = \frac{\lfloor n}{\lfloor r \lfloor n-r}$
5. $^nC_n = {}^nC_0 = 1$
6. $^nC_r = {}^nC_{n-r}$
7. If $^nC_x = {}^nC_y$, then $x = y$ or $x + y = n$
8. $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$
9. $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
10. $^nC_r : {}^{n-1}C_{r-1} = \frac{n}{r}$
11. $(x + a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + a^n$

    $(x - a)^n = x^n - {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + (-a)^n$
12. $(x + a)^n + (x - a)^n = 2$ [Sum of odd ordered terms]
13. $(x + a)^n - (x - a)^n = 2$ [Sum of even ordered terms]
14. $(1 \pm x)^n = 1 \pm {}^nC_1 x + {}^nC_2 x^2 \pm {}^nC_3 x^3 + \ldots + (\pm x)^n$
15. The general term in the expansion of $(x + a)^n$ is $T_{r+1} = {}^nC_r x^{n-r} a^r$

    The middle term in the expansion of $(x + a)^n$

    a. If n is odd , there are two middle terms of orders $\frac{n+1}{2}$ and $\frac{n+3}{2}$

    b. If n is even , there is one middle term of order $\frac{n+2}{2}$
16. In the expansion of $(x + a)^n$, The ratio between two consecutive terms $T_{r+1} : T_r = \frac{n-r+1}{r} \times \frac{a}{x}$
17. In the expansion of $(x + a)^n$, The ratio between the two coefficients of the two consecutive terms $\frac{c_0 . t_{r+1}}{c_0 . t_r} = \frac{n-r+1}{r} \times \frac{\text{coefficient of second term}}{\text{coefficient of first term}}$

General Exercises

First : Choose the correct answer :

1. ${}^{n}P_{r} : {}^{n-1}P_{r-1} =$

   a) n b) r c) $\frac{n}{r}$ d) $\frac{r}{n}$

2. If $|\underline{x - 3} \times {}^{x}P_{3} = 20|\underline{x - 2}$, then x =

   a) Zero b) 2 c) 5 d) 4

3. If ${}^{14}C_{r2} = {}^{14}C_{r+2}$, then r =

   a) 2 b) 4 c) 3 d) 2 or 3

4. ${}^{n}P_{r} \div {}^{n}P_{r-1} =$

   a) $n - r$ b) $n - r - 1$ c) $n - r + 1$ d) $n + r$

5. The expression ${}^{n}C_{r} + {}^{n}C_{r+1}$

   a) ${}^{n+1}C_{r}$ b) ${}^{n}C_{r+1}$ c) ${}^{n+1}C_{r+1}$ d) ${}^{n+1}C_{r+2}$

6. If ${}^{9}C_{r} > {}^{9}C_{r-1}$, then

   a) $r < 4$ b) $r > 4$ c) $r < 5$ d) $r > 5$

7. If ${}^{x+y}P_{2} = 210$, ${}^{y-3}C_{3} = 35$, then $|\underline{2x - y} =$

   a) 5 b) 10 c) 2 d) 1

8. If ${}^{7}C_{r} > 1$, ${}^{r}C_{5} > 1$, then the value of $|\underline{6 - r} =$ ....

   a) Zero b) 1 c) 720 d) 6

9. In the expansion of $(1 + x)^{n} = 1 + a_{1}x + a_{2}x^{2} + a_{3}x^{3} + a_{4}x^{4} + \ldots\ldots + a_{10}x^{10}$

   and $\frac{a_{2} + a_{3}}{a_{2}} = 3$, then n = ...

   a) 4 b) 6 c) 8 d) 9

10. If $1 + \frac{5}{2}x + \frac{5 \times 4}{4|\underline{2}}x^{2} + \frac{5 \times 4 \times 3}{8|\underline{2}}x^{3} + \ldots\ldots + \frac{1}{32}x^{5} = 1024$, then x = ...

    a) 5 b) 4 c) 6 d) 8

11. In the expansion of $(ax + b)^{2n+1}$. If the two middle terms are equal at $x = 2$, then : ....

    a) $a = 2b$ b) $b = 2a$ c) $ab = 2$ d) $ab = \frac{1}{2}$

Second : Answer the following :

12 If $^6P_r = 4 \times {}^6P_{r-1}$, $\underline{|n-r} = 6$, find n , r

13 If $^nC_{n-2} = 45$, $^mP_{m-3} = 4$, find the value of $^nC_{m-2} \div {}^nC_{m-1}$

14 If n (n - 1) (n - 2) .... (n - r + 1) = 5040 ,^nCr = 210, find the value of $^{n+1}C_{2r}$.

15 If $^nC_3 = 120$,$^nC_{r2+2r} = {}^nC_{2r+5}$, find the value of $^nC_{7r+3}$

16 If $^nC_r \div {}^nC_{r-1} = 5 : 7$,$^nP_r \div {}^nP_{r-1} = 5$, find the value of n and r.

17 If $^nC_r : {}^nC_{r-1} : 5\ {}^{n-1}C_{r-2} = 7 : 4 : 6$, find the value of n and r.

18 If we have the digits 1, 2, 3, 4, 5 , find how many even number greater than 300 can be formed from these digits, if the digits:

a Can be repeated

b Can not be repeated

19 If $\underline{|n} = 720$, $^{n+1}C_r : {}^{n+1}C_{r-1} = \frac{3}{5}$ find the value of $^{n+1}C_{r-2}$

20 If $^{n+2}C_{r+1} = 9 \times {}^{n+1}C_r = 90 \times {}^nC_{r-1}$, find the value of n, r.

21 If $^nP_r = 120 \times {}^nC_r$, find the value of $^{12}C_{2r}$ then find the least value of n

22 **Prove that :** $^nC_1 + (2 \times {}^nC_2) + (3 \times {}^nC_3) + (4 \times {}^nC_4) + ... + (n \times {}^nC_n) = n \times 2^{n-1}$

23 If $(1 + x)^{10} = 1 + {}^{10}C_1x + {}^{10}C_2 x^2 + ... + {}^{10}C_{10} x^{10}$, use that to find:

a $1 + {}^{10}C_1 + {}^{10}C_2 + ... + {}^{10}C_{10}$

b $1 - {}^{10}C_1 + {}^{10}C_2 - .... + {}^{10}C_{10}$

c $1 + 3 \times {}^{10}C_1 + 9 \times {}^{10}C_2 + 27 \times {}^{10}C_3 + ... + (3)^{10}$

24 In the expansion of $(x^2 + \frac{1}{x})^n$ according to the descending powers of x. If the coefficient of the fourth term equals the coefficient of the thirteenth term, find the value of n , then find the order and the value of the term free of x

25 In the expansion of $(1 + x)^n$ if $(T_3)^2 = T_2 \times T_5$, find the value of n when $x = \frac{9}{5}$

26 In the expansion of $(1 + x)^n$, if the coefficient of T_{10} is the arithmetic mean between the coefficient of T_9 , and the coefficient of T_{11} , find each of:

a n

b $^nP_3$

c $\underline{|n-19}$

27 In the expansion of $(1 + x)^n$ according to the ascending powers of x . If $T_8 : T_7 : T_6 = 6 : 14 : 21$, find the value of each of n and x.

28 If the ratio among three consecutive terms in the expansion of $(x + \frac{k}{x^2})^{27}$ is 15 : 6 : 2 where $k \in Z^+$, find the orders of these terms, then find the order and the value of the term free of x in this expansion.

29 In the expansion of $(1 + mx)^n$ according to the ascending power of x , if the second and the third terms are $\frac{-10}{3}x$, $5x^2$ respectively, find the value of m , n then find the value of the middle term of this expansion at $x = 3$

30 If the order of the term free of x in the expansion of $(2x^2 - \frac{3}{x})^{21}$ is equal to the order of the term free of x in the expansion of $(x + \frac{1}{x})^{2n}$, find the value of n then find the ratio between the two middle terms of the first expansion at x = -1

31 In the expansion of $(4x^2 + \frac{1}{2x})^{13}$, find the coefficient of x^5 then find the value of x which makes the two middle terms in this expansion equal, and prove that this expansion has no term free of x

32 In the expansion of $(1 + x)^n$ according to the ascending powers of x , if the coefficients of three consecutive terms are 15, 24, 28 respectively, find the value of n and the order of these terms

33 If the middle term of the expansion of $(1 + x)^{10}$ is equal to twice the seventh term, find the value of x

34 If the expansion of $(x^2 + \frac{1}{x})^n$ has a term free of x, prove that n is multiple of 3 , then find this term when n = 12.

35 If the two middle terms in the expansion of $(2x + 3)^{17}$ are equal , find the value of x.

36 If a , b are the two middle terms in the expansion of $(x - \frac{1}{x})^{15}$ according to the descending powers of x, prove that $a + b x^2 = 0$

37 If the ratio between the coefficient of the sixth term and the coefficient of the fourth term in the expansion of $(\frac{3}{2} + \frac{2x}{3})^n$ according to the ascending powers of x is equal to 8 : 27, find the value of n.

38 In the expansion of $(2x^2 + \frac{1}{2})^7$ according to the descending powers of x, if the third and the sixth terms are equal, find the value of x.

39 In the expansion of $(1 + x)^n$ according to the ascending powers of x. If $T_4 = \frac{25}{3} T_2$, $T_5 = T_6$, find the value of each of n, x

40 If n is a positive integer and $(1 + kx)^n = 1 + m_1 x + m_2 x^2 + m_3 x^3 + m_4 x^4 + ..$, $m_1 = 12$, $m_4 = 4m_2$, find the values of n, k.

41 If the third, fourth and fifth terms in the expansion of $(x + y)^n$ according to the descending powers of x are 112 , 448 , 1120, find the value of each of x, y , n.

42 In the expansion of $(\frac{2x}{3} + \frac{3}{2x^2})^{12}$, find each of the middle term and the term containing x^3

43 Find the term free of x in the expansion of $(x + \frac{1}{x})^6 - (x - \frac{1}{x})^6$

44. In the expansion of $(2x + \frac{3}{x^2})^n$ according to the descending powers of x , if the ninth and the tenth terms are equal , and the ratio between the sixth and the seventh terms is 8 : 15, find the value of n, and prove that the expansion has no term free of x.

45. In the expansion of $(x^2 + \frac{c}{x^3})^{15}$, find the value of c which makes the coefficient of x^{10} be equal to double the coefficient of x^{15}

46. In the expansion of $(x^2 + \frac{1}{x})^{15}$, find the ratio between the term free of x and the sum of the coefficients of the two middle terms.

47. In the expansion of $(x^k + \frac{1}{x})^8$ where k is the a positive integer, find :
   - **a** The values of k which make the expansion have a term free of x
   - **b** The ratio between the term free of x and the coefficient of the middle term of the greatest value of k which you obtained in (a).

48. If : $(1 + x)^n = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$,prove that :
   - **a** $\frac{a_1}{a_0} + \frac{2a_2}{a_1} + \frac{3a_3}{a_2} + \dots + \frac{{}^n a_n}{a_{n-1}} = \frac{n}{2}(n + 1)$
   - **b** $a_0 + 2 a_1 + 4 a_2 + + 8 a_3 + \dots + 2^n a_n = 3^n$

49. If the third term in the expansion of $(2x + \frac{1}{x^2})^n$ according to the descending powers of x is free of x, find the value of y which makes this term equal to the second term in the expansion of $(1 + y^3)^{30}$ according to the ascending powers of y.

50. In the expansion of $(x^2 + \frac{1}{ax})8$, if the coefficient of the middle term equals the coefficient of the term containing x^{10} , find the value of a.

51. In the expansion of $(x^2 - \frac{1}{x})^{14}$, prove that there is no term free of x, then find the ratio between the seventh term and the sixth term when x = -1

52. In the expansion of $(9x^2 + \frac{1}{3x})^9$, find the value of the term free of x, then prove that the two middle terms are equal when $x = \frac{1}{3}$

53. In the expansion of $(x^2 + \frac{1}{x})^9$, find the term free of x , and if the ratio between the term free of x and the sixth term is 9 : 4, find the real value of x

54. In the expansion of $(x^2 + \frac{1}{x})^{3n}$, find the coefficient of x^{3n} , and if n = 6, find the ratio between the coefficient of x^{3n} and the coefficient of the middle term.

For more activities and exercise, visit www.sec3mathematics.com.eg

Accumulative test

1. In the expansion of $(1 + x)^{17}$. If the coefficient of T_{r+4} is equal to the coefficient of T_{3r+3}, then r = ...

   a) 3 b) 4 c) 17 d) 7

2. In the expansion of $(1 + x)^{27}$. If the ratio between the two middle terms = 1 : 3, then x = ...

   a) 4 b) 3 c) $\frac{1}{3}$ d) $\frac{1}{4}$

3. The expression $(\sqrt{2} + 1)^5 - (\sqrt{2} - 1)^5$ = ...

   a) -82 b) 82 c) $58\sqrt{2}$ d) $-58\sqrt{2}$

4. The fourth term in the expansion of $(\frac{x}{3} + \frac{3}{x})^6$ is:

   a) 20 x b) $\frac{20}{x}$ c) 20 d) $20x^2$

5. The last term in the expansion of $(2 - x)^5 (2 + x)^5$ is ....

   a) x^5 b) $-x^5$ c) $-x^{10}$ d) x^{10}

6. In the expansion of $(1 + x)^n$ prove that $\frac{T_{r+1}}{T_r} = \frac{n - r + 1}{r}x$. If the coefficient of T_{13} according to the ascending powers of x is equal to the coefficient of T_{14}, find the value of n and if $\frac{T_7}{T_5} = \frac{7}{8}$, find the value of x

7. a) If ${}^nC_2 + {}^nC_3 = n^2 + 6n + 5$, find the value of n .

   b) Find the value of the term free of x in the expansion of $(\frac{1}{x^2})(x + \frac{1}{x^3})^{10}$

8. In the expansion of $(1 - mx)^n$ according to the ascending powers of x.
   If the second term = $\frac{-1}{4}x$ and the third term = $\frac{3}{100}x^2$, find the value of each of m, n.

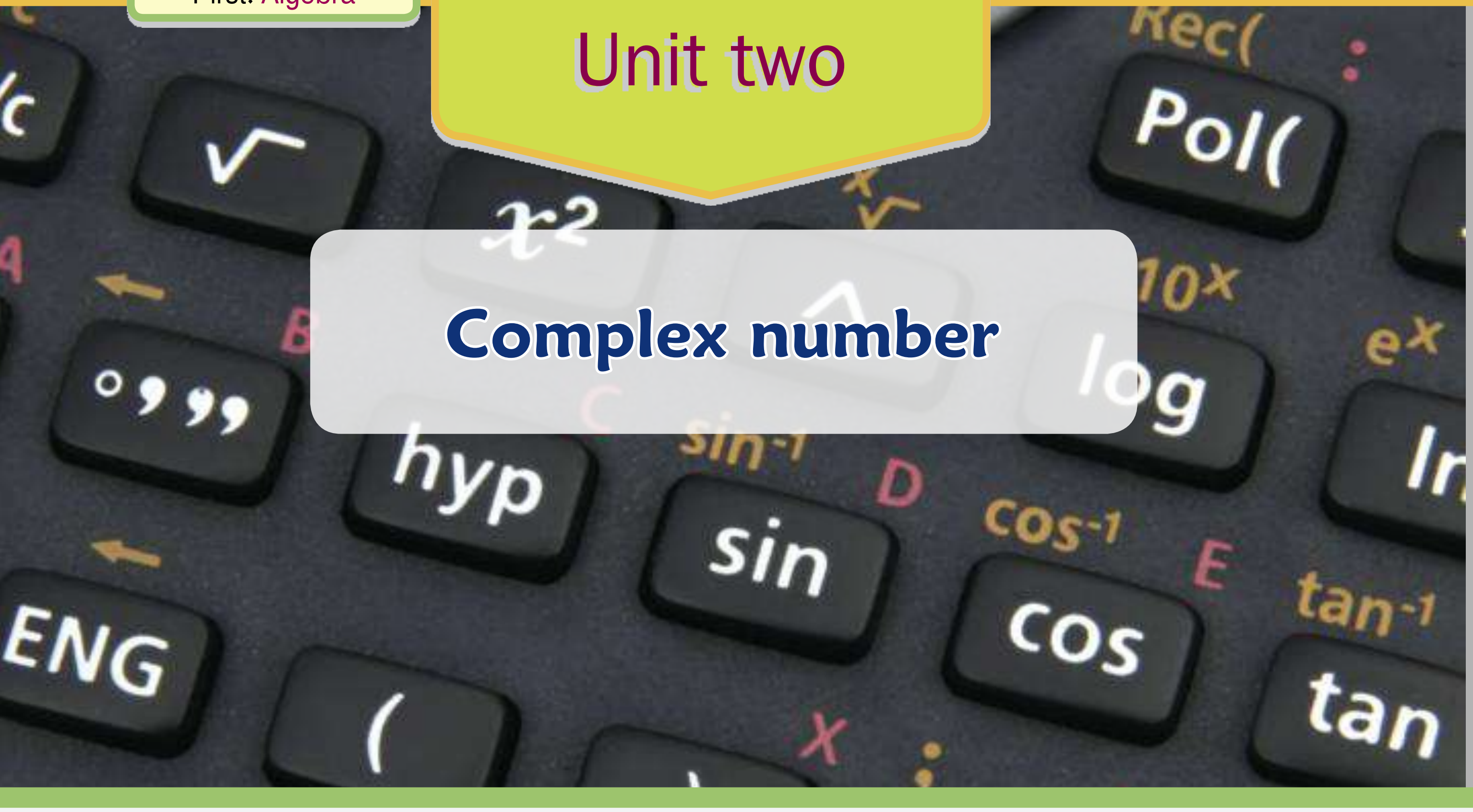
9. How many numbers of value less than 400 can be formed from the digits {1, 2, 3, 4, 5} in each of the following.
   First: the digits can be repeated **Second**: the digits can not be repeated.

10. If the ratio between the fifth , sixth and seventh terms in the expansion of $(\frac{3x}{2} + \frac{2}{3x})^n$ according to the descending powers of x is 40 : 24 : 11 , find the value of n, x

First: Algebra

Unit two

Complex number



Unit introduction

Jean-Robert Argand is one of the most popular mathematicians. He had been the first to study the complex numbers in details and had used them to prove that all the algebraic equations have roots whether these roots are true or imaginary. The complex numbers are represented by the Argand's Diagram to honor the French Scientist Argand either by point (x, y) where x is a real number on x-axis and y represents the imaginary number on y-axis or vector of amagintude $\sqrt{x^2 + y^2}$ and direction $\tan^{-1}\frac{y}{x}$. in this unit, you are going to identify the cubic roots of unity and solve applications on the complex numbers such as electricity, dynamics, theorem of relativity and the physical different fields. These numbers are flexiable to help to get a final result satisfactorily

Unit objectives

By the end of this unit and doing all the activities included, the student should be able to:

- Represent the complex number and its conjugate graphically by points (ordered pairs) in the cartesian plane.
- Determine the modulus and the amplitude of the complex number.
- Identify the principle amplitude of a complex number
- Identify the trigonometric from of a complex number
- Identify De Moivre's theorem and its applications
- Deduce the n^{th} roots of any complex number
- Express sin ni , cos ni interms of the powers of sini , cosi
- Identify the expansion of sini , cosi, as series
- Deduce the Eular's rule from the series
- Identify and apply the methods to convert the different forms of the complex number.
- Identify the cubic roots of unity.
- Identify the modulus and the amplitude of the product and quotient of two complex numbers.
- Perform the basic operations on the complex numbers in the trigonometric form
- Solve Application problems on the cubic roots of unity
- Use the complex numbers in solving Mathematical problems.
- Use some computer programs in solving mathematical problems including complex numbers
- Deduce the properties of addition and multiplication operations on the complex numbers.
- Deduce the properties of two conjugate numbers
- Deduce the properties of the cubic roots of unity.

Key terms

- Argand plane
- conjugate
- Modulus
- principle amplitude
- Trigonmetric
- De Moivre's thearem
- root
- square root
- cubic root
- Unit circle
- Polar

Materials

- Scientific calculator

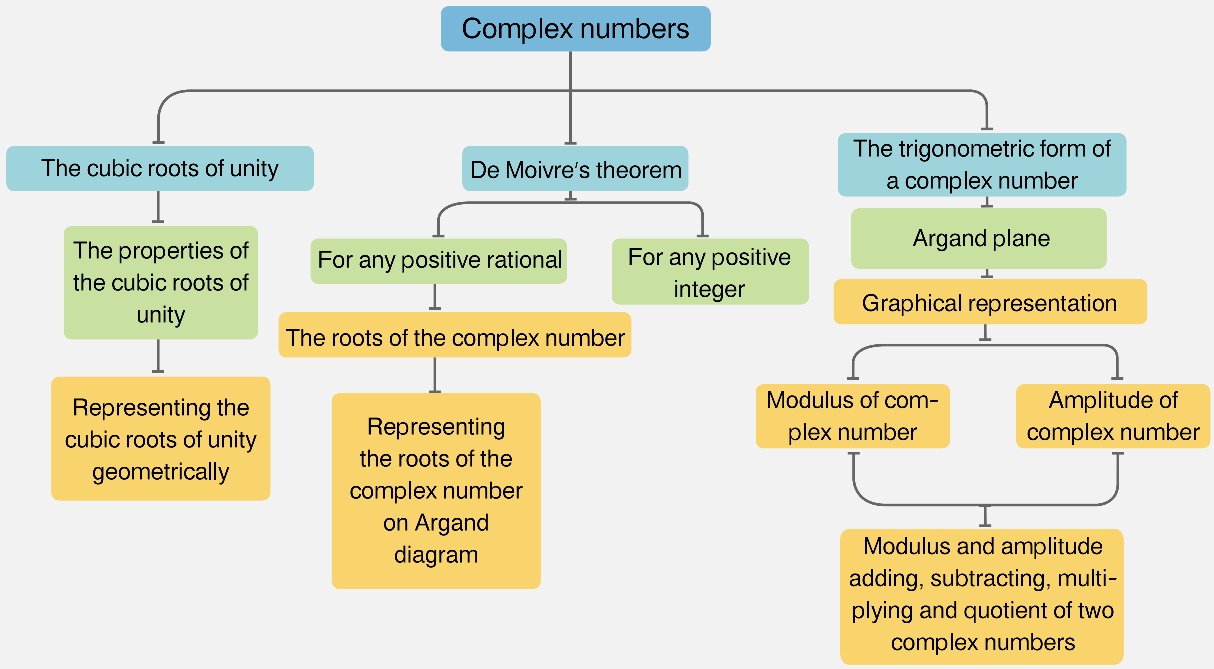
Unit Lessons

Lesson (2-1): The trigonometric form of a complex number

Lesson (2-2): De Moivre's theorem

Lesson (2-3): The cubic roots of unity

Unit chart



Unit two

2 - 1 Trigonometric form of a complex number

You will learn

- The graphical representation of the complex number and its conjugate in Argand's plane
- Graphical representation of the sum of two complex numbers
- Modulus of the complex number
- Amplitude of the complex number
- Principle amplitude of the complex number
- The trigonometric form of the complex number
- The modulus and the amplitude of the product and quotient of two complex numbers.

Key terms

- Argand's plane
- Conjugate
- Modulus
- Principle amplitude
- Trigonmetric form

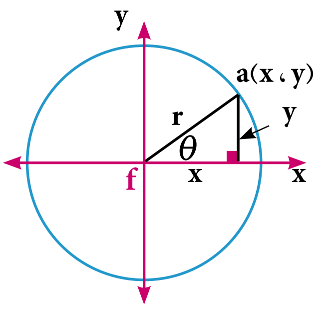
Materials

- Scientific calculator

You have studied the complex numbers and known that a complex number can be written in the form $z = x + y\,i$ **(Algebraic form),** where x , y are real numbers, $i^2 = -1$. In this lesson, we will identify another form to write the complex number and how to represent it graphically .

Polar and cartesian coordinates:

The opposite figure represents a circle with radius length r, A (x , y) lies on the circle and opposite to an angle of measure θ .



$\cos\theta = \frac{x}{r}$ $\qquad \sin\theta = \frac{y}{r}$

$x = r\cos\theta$ $\qquad y = r\sin\theta$

where $r = \sqrt{x^2 + y^2}$, $\tan\theta = \frac{y}{x}$

i.e. : $\theta = \tan^{-1}\frac{y}{x}$ if we meditate the cartesian plane as a polar plane where the polar axis coincides on the positive part of x-axis, then we can change polar coordinates to Cartesian and vise versa.

Converting polar co-ordinates to cartisian co-ordinates

If the point A in polar coordinates is A(r, θ), then the Cartesian coordinate of A is (x, y) where :

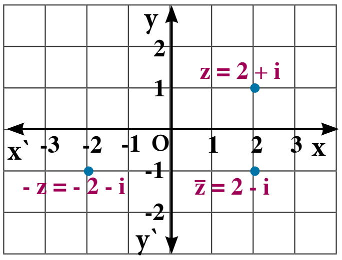
$x = r\cos\theta$, $y = r\sin\theta$

Then: $(x, y) = (r\cos\theta , r\sin\theta)$

Argand's plane

The mathematician **Argand** had represented the complex number z graphically on the orthogonal Cartesian coordinates such that the axis $\overleftrightarrow{xx`}$ represents the real part of the complex number and the axis $\overleftrightarrow{yy`}$ represents the imaginary part of the complex number. so the point (x , y)represents the complex number x + i y

Example



1. In the opposite Argand's diagram, we notice that the two points representing z , - z are symmetric about the point of origin (O).

We also notice that the two points representing the conjuget number z, $\bar{z}$ are symmetric about $\overleftrightarrow{x\,x^-}$ axis

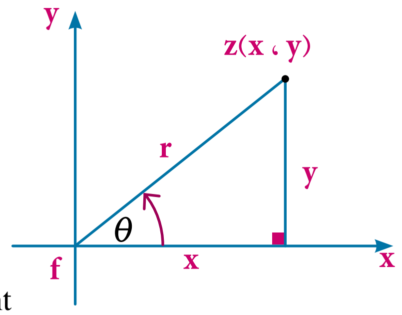
Try to solve

1 Represent on Argend's diagram each of the numbers:

$z = 3 + 4i$, $\bar{z}$, $-z$, $1 + z$

Critical thinking: What do all complex number with real part equals 2 represent on Argand's diagram?

Learn



Where $\frac{-\pi}{2} < \tan^{-1}(\frac{y}{x}) = \frac{\pi}{2}$

The modulus and the amplitude (argument) of a complex number

If $z = x + y\,i$ is a complex number represented by the point z(x , y) in Argand's plane. Then the modulus is its distance from the origin and denoted by |z| or r and θ is called amplitude of the complex number:

$r = \sqrt{x^2 + y^2}$, $\tan\theta = \frac{y}{x}$ thus $\theta = \tan^{-1}\frac{y}{x}$

The trigonometric (polar) form of a complex number

If $z = x + yi$ is a complex number with modulus r and principle amplitude θ where $\theta \in]-\pi, \pi]$, then it is written as $z = r(\cos\theta + i\sin\theta)$ and the measure of θ is determined according to the following:

a $x > 0, y > 0$ θ lies in the first quadrant $\theta = \tan^{-1}(\frac{y}{x})$

b $x < 0, y > 0$ θ lies in the second quadrant $\theta = \pi + \tan^{-1}(\frac{y}{x})$

c $x < 0, y < 0$ θ lies in the third quadrant $\theta = -\pi + \tan^{-1}(\frac{y}{x})$

d $x > 0, y < 0$ θ lies in the fourth quadrant $\theta = \tan^{-1}(\frac{y}{x})$

Notic that

$x > 0, y = 0$, then $\theta = 0$

$x < 0, y = 0$, then $\theta = \pi$

$x = 0, y > 0$, then $\theta = \frac{\pi}{2}$

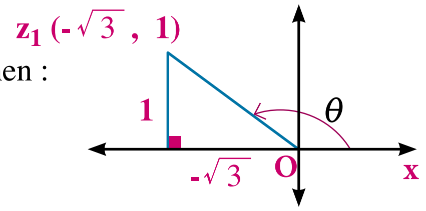
$x = 0, y < 0$ then $\theta = \frac{-\pi}{2}$

Example

2 Find the modulus and the principle amplitude of each of the following complex numbers:

a $z_1 = -\sqrt{3} + i$ **b** $z_2 = -1 - i$

Solution



$\because$ The form of the complex number is : $z = x + i\,y$, then :

a $x = -\sqrt{3}$, $y = 1$

$\therefore$ z_1 lies in the second quadrant

$r = \sqrt{x^2 + y^2} = \sqrt{(-\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$

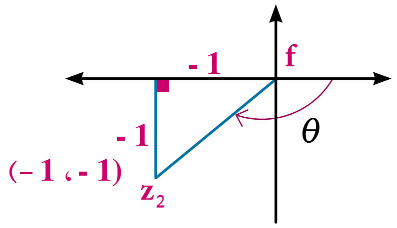
$\theta = \pi + \tan^{-1}(\frac{-1}{\sqrt{3}}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

b $x = -1 \quad , \quad y = -1$

$\therefore$ z_2 lies in third quadrant

$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = -\pi + \tan^{-1}(\frac{-1}{-1}) = -\pi + \frac{\pi}{4} = \frac{-3\pi}{4}$



Try to solve

2 Find the modulus and the principle amplitude of each of the following:

a $z_1 = \sqrt{2} + \sqrt{2}\, i$ **b** $z_2 = 1 - \sqrt{3}\, i$

c $z_3 = -\sqrt{3}\, i$ **d** $z_4 = 5$

Remember

$\frac{\theta^\circ}{180^\circ} = \frac{\theta^{rad}}{\pi}$

This rule is used to convert from degree measure into radian measure and vise versa

The properties of the modulus and the amplitude of a complex number

For each complex number $z = x + y\, i$ with argument θ :

1) $|z| \geqslant 0$

2) The amplitude of the complex number takes an infinite number of values by adding the multiple of 2π. Thus the amplitude of the complex number is $\theta + 2\pi n$ where $n \in Z$.

3) $|z| = |\bar{z}| = |-z| = |\overline{-z}|$

4) $z\,\bar{z} = |z|^2$

Critical thinking: If the principle amplitude of Z is θ , find the principle amplitude of $-Z$, $\bar{z}$, $\frac{1}{Z}$

Example

3 Write each of the following complex numbers in a trigonometric form:

a $Z_1 = 2 - 2\sqrt{3}\, i$ **b** $Z_2 = -4i$ **c** $Z_3 = \frac{-4}{\sqrt{3} + i}$ **d** $Z_4 = -2$

Solution

$\because$ The form of the complex number is : $Z = x + i\,y$, then:

a $x = 2 \quad , \quad y = -2\sqrt{3}$

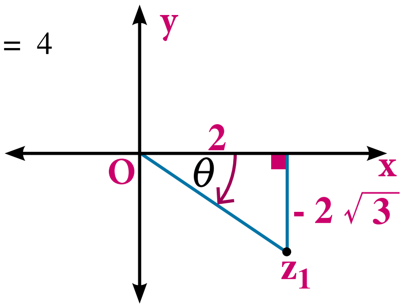
$\therefore Z_1$ lies in the fourth quadrant

$r = |Z_1| = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (-2\sqrt{3})^2} = \sqrt{16} = 4$

$\theta = \tan^{-1}(\frac{-2\sqrt{3}}{2}) = \frac{-\pi}{3}$

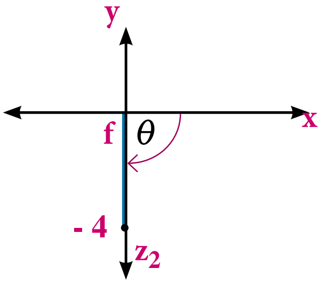
$\therefore Z_1 = r(\cos\theta + i\sin\theta)$

$= 4(\cos(\frac{-\pi}{3}) + i\sin(\frac{-\pi}{3}))$



b $\because$ x = 0 , y = - 4

$\therefore$ Z_2 lies on y- axis

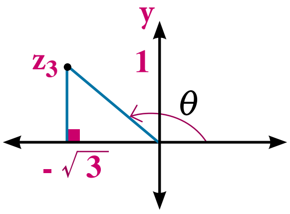


$r = |Z_2| = \sqrt{x^2 + y^2} = \sqrt{(0)^2 + (-4)^2} = 4$

$\theta = \frac{-\pi}{2}$ $\quad Z_2 = 4(\cos(\frac{-\pi}{2}) + i\sin(\frac{-\pi}{2}))$

c $Z_3 = \frac{-4}{\sqrt{3} + i} \times \frac{\sqrt{3} - i}{\sqrt{3} - i} = -\sqrt{3} + i$

$x = -\sqrt{3}$, y = 1 $\quad\therefore$ Z_3 lies in the second quadrant



$r = |Z_3| = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2$

$\theta = \pi + \tan^{-1}(\frac{I}{\sqrt{3}}) = \frac{5\pi}{6}$

$Z_3 = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$

d $\because$ x = - 3 , y = 0 $\quad\therefore$ Z_4 lies on x-axis

$r = |Z| = \sqrt{(-3)^2 + (0)^2} = 3$

$\theta = \pi$ $\quad\therefore$ $Z_4 = 3(\cos\pi + i\sin\pi)$

Remember

- $1 = \cos 0 + i\sin 0$
- -1 = cos r+ i sin r
- $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
- $-i = \cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}$

Try to solve

3 Write each of the following in a trigonometric form:

a $Z_1 = 8$ **b** $Z_2 = 5i$ **c** $Z_3 = -3 - 3i$

Example

4 Find the modulus and the principle amplitude for each of the following numbers:

a $Z_1 = -8(\cos 45^\circ + i\sin 45^\circ)$

b $Z_2 = 2(\sin\frac{4}{3}\pi - i\cos\frac{4}{3}\pi)$

Remember

$\cos\theta - i\sin\theta$

$= \cos(-\theta) + i\sin(-\theta)$

Solution

a $Z_1 = -8(\cos 45^\circ + i\sin 45^\circ)$

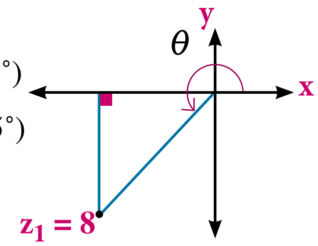
$= 8(-\cos 45^\circ - i\sin 45^\circ)$

$\because$ x < 0 , y < 0 $\quad\therefore$ Z_1 lies in the third quadrant

$\because$ $-\cos 45^\circ = \cos(180^\circ + 45^\circ)$, $-\sin 45^\circ = \sin(180^\circ + 45^\circ)$

$\therefore$ $Z_1 = 8(\cos 225^\circ + i\sin 2258 = (^\circ(\cos - 135^\circ + i\sin - 135^\circ)$

$\therefore$ $Z_1 = 8$, principle amplitude $\theta = -135^\circ$



b $Z_2 = 2\,(\sin \frac{4}{3}\pi - i \cos \frac{4}{3}\pi)$

$\because\ x > 0\ ,\ \ y < 0$

$\because\ \sin \frac{4}{3}\pi = \cos(\frac{3\pi}{2} + \frac{4}{3}\pi) = \cos \frac{17}{6}\pi = \cos(\frac{5\pi}{6})$

$-\cos \frac{4}{3}\pi = \sin(\frac{3\pi}{2} + \frac{4}{3}\pi) = \sin(\frac{17}{6}\pi) = \sin(\frac{5\pi}{6})$

$\therefore\ z_2 = 2(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})$

$\therefore\ |z_2| = 2$, principle amplitude $\frac{5\pi}{6}$

Remember

$\sin(\frac{\pi}{2} + \theta) = \cos\theta$

$\cos(\frac{\pi}{2} + \theta) = -\sin\theta$

$\sin(\frac{3\pi}{2} + \theta) = -\cos\theta$

$\cos(\frac{3\pi}{2} + \theta) = \sin\theta$

Try to solve

4) Find the modulus and the principle argument for each of the following complex numbers:

a $Z_1 = 2\,(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3})$ **b** $Z_2 = \frac{-1}{\sqrt{2}}(\sin 45° - i \sin 45°)$

Learn

multiplying and dividing complex numbers using the trigonometric form

If $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then

1) $Z_1Z_2 = r_1r_2\,(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2\,(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$ **(1)**

Thus. $|Z_1Z_2| = r_1r_2 = |Z_1||Z_2|$

$\text{Arg}\,(Z_1Z_2) = \theta_1 + \theta_2$

2) $\frac{Z_1}{Z_2} = \frac{r_1}{r_2} \times \frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$ **(2)**

then $|\frac{Z_1}{Z_2}| = \frac{r_1}{r_2} = \frac{|Z_1|}{|Z_2|}$, $\text{Arg}\,(\frac{Z_1}{Z_2}) = \theta_1 - \theta_2$

Ask your teacher to prove the relations (1) and (2)

Example

5) Express $3\,(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}) \times 4\,(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$ in the form of $x + y\,i$

Solution

$3\,(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}) \times 4(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$

$= 3 \times 4((\cos\frac{5\pi}{12} + \frac{\pi}{12}) + i\sin(\frac{5\pi}{12} + \frac{\pi}{12}))$

$= 12\,(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 12(0 + i\,(1)) = 12\,i$

Try to solve

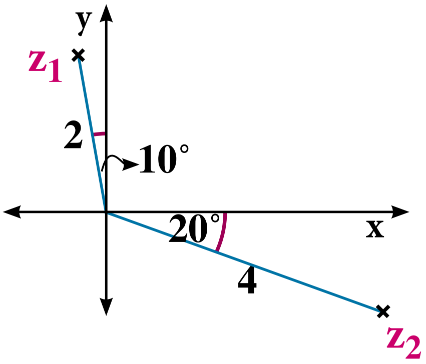
5) Express $2(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15}) \times 3(\cos\frac{-2\pi}{5} + i\sin\frac{-2\pi}{5})$ in the form of $x + iy$

Remember

- $1 = \cos 0 + i\sin 0$
- $-1 = \cos\pi + i\sin\pi$
- $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
- $-i = \cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}$

Example

6) If Z_1 , Z_2 are two complex numbers represented on Argend's plane as in the opposite figure, find in the form of $x + yi$ the number $\frac{Z_2}{Z_1}$



Solution

From the graph $|Z_1| = 2$, arg of $Z_1 = 90^\circ + 10^\circ = 100^\circ$

$\therefore Z_1 = 2(\cos 100^\circ + i\sin 100^\circ)$

$|Z_2| = 4$, arg $Z_2 = -20^\circ$

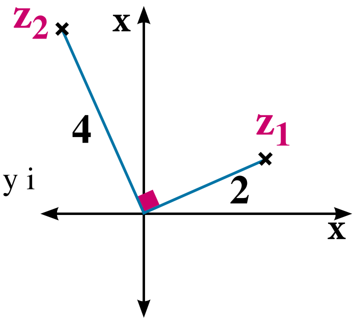
$\therefore z_2 = 4(\cos(-20^\circ) + i\sin(-20^\circ))$

$\therefore \frac{Z_2}{Z_1} = \frac{4}{2} \times \frac{\cos(-20^\circ) + i\sin(-20^\circ)}{(\cos 100^\circ + i\sin 100^\circ)}$

$= 2[\cos(-20^\circ - 100^\circ) + i\sin(-20^\circ - 100^\circ)]$

$= 2(\cos(-120^\circ) + i\sin(-120^\circ))$

$= 2(-\frac{1}{2} - i \times \frac{\sqrt{3}}{2}) = -1 - \sqrt{3}\, i$



Try to solve

6) Use the opposite Argand's diagram to find $\frac{Z_2}{Z_1}$ in the form $x + yi$

Results:

1) If $Z = r(\cos\theta + r\sin\theta)$ then

(a) $\frac{1}{Z} = \frac{1}{r}(\cos(-\theta) + r\sin(-\theta))$ (b) $Z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

2) We can generalize the product of finite number of complex numbers if $Z_1, Z_2, \ldots Z_n$ are complex numbers and if:

$Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, ... ,$Z_n = r_n(\cos\theta_n + i\sin\theta_n)$

then: $Z_1 Z_2 \ldots\ldots Z_n = r_1 r_2 \ldots\ldots r_n(\cos(\theta_1 + \theta_2 + \ldots\ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots\ldots + \theta_n))$

in a special case when $Z_1 = Z_2 = \ldots\ldots = Z_n = r(\cos\theta + i\sin\theta)$ then:

$$Z^n = r^n(\cos n\theta + i\sin n\theta)$$

Example

7) Put the number $1 - i$ in the trigonometric form, then find $(1 - i)^8$

Solution

$\because\ x = 1 \quad , \quad y = -1 \qquad r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\because\ x > 0\ ,\ y < 0 \qquad \therefore$ Z lies in the fourth quadrant

$\theta = \tan^{-1}(\frac{-1}{1}) = \frac{-\pi}{4} \qquad \therefore\ 1 - i = \sqrt{2}\,(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4})$

$\therefore\ (1 - i)^8 = (\sqrt{2})^8\ (\cos(\frac{-\pi}{4}) + i\sin(\frac{-\pi}{4}))^8$

$= 16\,(\cos(-2\pi) + i\sin(-2\pi))$

$= 16\,(\cos 0 + i\sin 0) = 16$

Try to solve

7) If $Z_1 = 2(\cos 10^\circ + i\sin 10^\circ)$, $Z_2 = 3(\cos 40^\circ + i\sin 40^\circ)$
Find $Z_1^4\ Z_2^2$ in the form $x + y\,i$

Exponential form of a complex number (Euler form)

Any function of x can be expressed as a series from the power of x called (Taylor series)
Here are Taylor expansion for some functions that we are going to study in this unit.

1) The sine function $y = \sin x$

$$\sin x = \frac{x}{\lfloor 1} - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots\ldots + (-1)^n \times \frac{x^{2n+1}}{\lfloor 2n+1} + \ldots\ldots$$

(Sine function is an odd function because $\sin(-x) = -\sin x$ so the expansion contains the odd power of x)

2) The cosine function $y = \cos x$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots\ldots + (-1)^n \times \frac{x^{2n}}{\lfloor 2n} + \ldots\ldots$$

(cosine function is an even function because $\cos(-x) = \cos x$ so the expansion contains even powers of x)

3) exponential function $y = e^x$

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} \ldots\ldots + \frac{x^n}{\lfloor n} + \ldots\ldots$$

Notice that

$$e^{ix} = 1 + \frac{ix}{\lfloor 1} + \frac{i^2x^2}{\lfloor 2} + \frac{i^3x^3}{\lfloor 3} + \ldots\ldots + \frac{i^nx^n}{\lfloor n} + \ldots\ldots$$

$$= 1 + \frac{ix}{\lfloor 1} - \frac{x^2}{\lfloor 2} - \frac{ix^3}{\lfloor 3} + \ldots\ldots$$

$$= (1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots\ldots) + i\,(\frac{x}{\lfloor 1} - \frac{x^3}{\lfloor 3} + \ldots\ldots)$$

$$e^{ix} = \cos x + i\sin x$$

i.e. The complex number $Z = x + i\,y = r(\cos\theta + i\sin\theta)$ can be written in the form :

$Z = r\,e^{\theta i}$ it called Euler form where θ is in radian measure.

Tip
Euler equation
$e^{\pi i} + 1 = 0$
it conects the most 5 famous constants in Euler form
θ should be in radian measure

Example

8 Write each of the following complex numbers in the exponential form (Euler's form):

a $Z_1 = 1 + i$ b $Z_2 = -1+\sqrt{3}\ i$ c $Z_3 = e^{3+\frac{\pi}{6}i}$ d $Z_4 = -2i$

Solution

a $Z_1 = 1 + i$ $\therefore x = 1 \ , \ y = 1$

$r = |Z_1| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

$\because x > 0 \ , \ y > 0$ $\therefore Z_1$ lies in the first quadrant

$\theta = \tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}$ $\therefore Z_1 = r e^{\theta i} = \sqrt{2}\ e^{\frac{\pi}{4}i}$

b $Z_2 = -1+\sqrt{3}\ i$ $\therefore x = -1 \ , \ y = \sqrt{3}$

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\because x < 0 \ , \ y > 0$ $\therefore Z_2$ lies in the second quadrant

$\therefore \theta = \pi + \tan^{-1}(-\sqrt{3}) = \frac{2}{3}\pi$ $\therefore Z_2 = r e^{\theta i} = 2 e^{\frac{2}{3}\pi i}$

c $Z_3 = e^{3+\frac{\pi}{6}i} = e^3 \times e^{\frac{\pi}{6}i}$ $r = |Z_3| = e^3 \ , \ \text{amp } Z_3 = \frac{\pi}{6}$

d $\because Z_4 = -2i$ $\therefore x = 0 \ , \ y = -2$ $\therefore r = \sqrt{(0)^2 + (-2)^2} = \sqrt{4} = 2$

$\because x = 0 \ , \ y = -2$ $\therefore Z_4$ lies on y-axis

$\therefore \theta = \frac{-\pi}{2}$ $\therefore Z_4 = 2 e^{\frac{-\pi}{2}i}$

Try to solve

8 If $Z = \frac{\sqrt{2}\ i}{1+i}$,write Z in the exponential form.

Multiplying and dividing the complex numbers using the exponential form.

If $z_1 = r_1 e^{\theta_1 i}$, $z_2 = r_2 e^{\theta_2 i}$

Then $z_1 z_2 = r_1 r_2 e^{\theta_1 i} \times e^{\theta_2 i} = r_1 r_2 e^{(\theta_1 + \theta_2)i}$,

$\frac{z_1}{z_2} = \frac{r_1}{r_2} \times \frac{e^{\theta_1 i}}{e^{\theta_2 i}} = \frac{r_1}{r_2} e^{(\theta_1 - \theta_2)i}$

Example

9 Find the result for each of the following in the exponential form :

a $2(\cos 25° + i \sin 25°) \times 2(\sin 158° - i \sin 158°)$ b $(\frac{1+i}{1-i})^7$

Solution

a Convert Z_2 to the standard trigonometric form as follows:

$\because$ (sin 158° - i cos 158°) = sin (90° + 68°) - i cos (90° + 68°) = cos 68° + i sin 68°

$\therefore$ 3 (cos 25° + i sin 25°) × 2 (cos 68° + i sin 68°)

= 6 (cos (25° + 68°) + i sin (25° + 68°))... sin 25°) ×2 (cos 68°.....

= 6 (cos 93° + i sin 93°) = $6\,e^{1,62i}$

b $\because 1 + i = \sqrt{2}\,(\cos 45° + i \sin 45°)$

$1 - i = \sqrt{2}\,(\cos(-45°) + i \sin(-45°))$

$\therefore \left(\frac{1+i}{1-i}\right) = \cos(45° + 45°) + i \sin(45° + 45°)$

$= \cos 90° + i \sin 90° = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$

$\therefore \left(\frac{1+i}{1-i}\right)^7 = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)^7 = \cos \frac{7\pi}{2} + i \sin \frac{7\pi}{2}$

$= \cos\left(\frac{-\pi}{2}\right) + i \sin\left(\frac{-\pi}{2}\right) = e^{\frac{-\pi}{2} i}$

Note

$\frac{93°}{180°} = \frac{\theta^E}{\pi}$

i.e: $\theta^d = \frac{93°}{180°} \times \pi$

$\theta^d \simeq 1,62$

Try to solve

9 If $Z_1 = 1 - \sqrt{3}\, i$, $Z_2 = 1 + i$, find each of the following in the trigonometrical form:

a $Z_1 Z_2$ **b** $\frac{Z_2}{Z_1}$ **c** $(Z_2)^6$

Example

10 Express $Z = \sqrt{2}\, e^{\frac{3\pi}{4} i}$ in the algebraic form x + y i where x , y ∈ R

Solution

$\because Z = \sqrt{2}\, e^{\frac{3\pi}{4} i}$ $\therefore r = |Z| = \sqrt{2}$, $\theta = \frac{3\pi}{4}$

$\therefore Z = \sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$

$= \sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$

$= -1 + i$

Note

$\cos \frac{3\pi}{4} = \cos 135° = \frac{-1}{\sqrt{2}}$

$\sin \frac{3\pi}{4} = \sin 135° = \frac{-1}{\sqrt{2}}$

Try to solve

10 Express $Z = 8\, e^{\frac{\pi}{6} i}$ in the algebraic form x + y i where x , y ∈ R

Exercises 2 - 1

Complete the following

1. The number $Z = 3 - 4i$ is represented on Argand's diagram by the point A where A = (.............. , ...)

2. If the point A represents the complex number Z on Argand's plane and point B represents the number $\bar{z}$ on Argand's plane, then B is the image of A by reflection in ....

3. The modulus of the complex number $Z = -5i$ equals ....

4. If $Z = \frac{2-i}{2+i}$ then $|Z| =$ ....

5. If θ is the principle argument of the complex number Z, then the principle arg of $\bar{z}$ is ....

6. If $Z = \frac{1}{\bar{z}}$, then $|Z| =$ ....

7. The exponential form of the number $-1 + i$ is ....

8. If $Z = 1 + \sqrt{3}\,i$, then the principle amplitude of Z^8 is ....

9. The trigonometric form of the number $Z = 2 - 2\sqrt{3}\,i$ is ....

10. If Z is a complex number and $\arg(z) = \theta$, then arg 2Z is ....

Choose the correct answer from the given:

11. If $Z = \sqrt{2}\,(\sin 30^\circ + i\cos 30^\circ)$, then the principle amplitude of Z is ....
    a) 30° b) 60° c) 90° d) 120°

12. If $Z = (1 + \sqrt{3}\,i)^n$ and $|Z| = 8$ then the principle amplitude of Z is ....
    a) $\frac{\pi}{2}$ b) $\frac{\pi}{3}$ c) $\frac{\pi}{6}$ d) π

13. If $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ and if $\theta_1 + \theta_2 = \pi$, then $Z_1 Z_2 =$ ....
    a) r_1r_2 b) $-r_1r_2$ c) r_1r_2i d) $-r_1r_2\,i$

14. The amplitude of the complex number $Z = -3$ is ....
    a) 0° b) 90° c) 180° d) 270°

15. If $Z = -1 + \sqrt{3}$, then $|\bar{z}| =$ ....
    a) $-1 - \sqrt{3}\,i$ b) $\sqrt{2}$ c) 2 d) -2

16 If $Z_1 = -1 - i$, then the exponential form of Z is ....

a $e^{\frac{3\pi}{4} i}$ b $e^{\frac{5\pi}{4} i}$ c $\sqrt{2}\, e^{\frac{-3\pi}{4} i}$ d $\sqrt{2}\, e^{225\, i}$

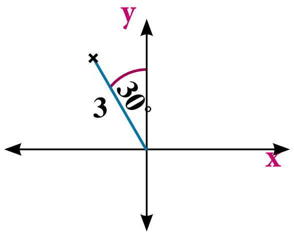
17 If $Z_1 = 2 + 2\sqrt{3}\, i$, $Z_2 = -3 - 3\sqrt{3}$ then arg $Z_1 + Z_2 =$ ....

a 60° b 240° c 180° d 300°

18 If $x + y\, i = \frac{a + b\, i}{a - b\, i}$, then $x^2 + y^2$ ....

a $a^2 + b^2$ b $a^2 - b^2$ c $2a\ b$ d 1

19 The opposite figure represents the complex number ....



a $3(\cos 30^\circ + i \sin 30^\circ)$

b $3(\cos 60^\circ + i \sin 60^\circ)$

c $3(\cos 120^\circ + i \sin 120^\circ)$

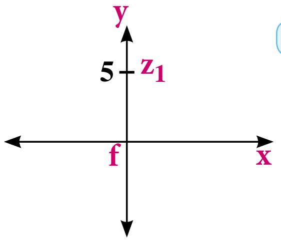
d $3(\cos 150^\circ + i \sin 150^\circ)$

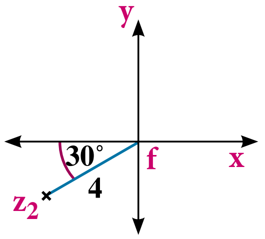
20 If Z is a complex number whose principle argument is θ, then the arg of $\frac{1}{Z}$ ....

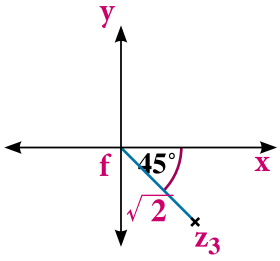
a θ b $-\theta$ c $\pi - \theta$ d $-\pi + \theta$

Answer the following:

21 Write each of the following complex numbers in the trigonometric form:

a


b


c


d $Z_4 = -3 + 4i$

e $Z_5 = 4(\cos 40^\circ - i \sin 40^\circ)$

22 Find the modulus and the principle argument for each of the following complex numbers:

a $Z_1 = -1 + i$

b $Z_2 = \frac{4}{\sqrt{3} - i}$

c $Z_3 = -2(\cos 45^\circ + i \sin 45^\circ)$

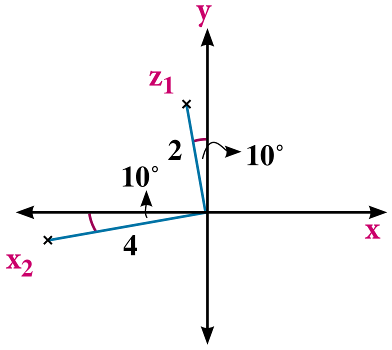
d $Z_4 = 1 + i \tan 20^\circ$

23. If $Z_1 = \cos 114^\circ + i \sin 66^\circ$, $Z_2 = \cos 42^\circ + i \sin 138^\circ$

$Z_3 = \sin 24 + i \sin 114$, find the algebraic form of the number: $\frac{Z_1 Z_2}{Z_3}$

24. If $Z_1 = 2 (\cos 75^\circ + i \sin 75^\circ)$, $Z_2 = 4 (\cos 15^\circ + i \sin 15^\circ)$, find the exponential form of: Z_1Z_2 , $\frac{Z_1}{Z_2}$

25. **In the opposite figure,** find the exponential form of: $\frac{Z_1}{Z_2}$



26. Write each of the following numbers in the algebraic form:

a. $Z_1 = e^{\frac{\pi}{3} i}$

b. $Z_2 = 2 e^{\frac{3\pi}{4} i}$

c. $3 e^{\frac{-\pi}{6} i}$

27. If $Z = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$, prove that $\frac{1}{z} = \frac{1}{2} e^{\frac{5\pi}{3} i}$

28. If $Z = \sqrt{3} + i$, find the algebraic form of Z^6

29. If $Z = \frac{(a + b) + i (a - b)}{(a - b) - i (a + b)}$, find Z in the simplest form then find $|Z|$ where $a, b \in R$

30. **Creative thinking:** If $Z_1 = \cos 75 + i \sin 75$, $Z_2 = \cos 15 + i \sin 15$, find the trigonometric form of the number: $Z_1 + Z_2$

31. If $\arg Z_1 = \frac{\pi}{3}$, $\arg Z_2 = \frac{3\pi}{4}$, $\arg Z_3 = \frac{\pi}{6}$, find:

a. $\arg (Z_1^3 \; Z_2^2)$

b. $\arg (2Z_1 \cdot Z_2)$

c. $\arg (\frac{Z_1Z_2}{Z_3})$

d. $\arg (Z_3^6)$

32. **Creative thinking:** Prove that $\cos \theta = \frac{1}{2} (e^{\theta i} + e^{-\theta i})$, $\sin \theta = \frac{-i}{2} (e^{\theta i} - e^{-\theta i})$

Unit two

2 - 2

Demoivre's Theorem

You will learn

- *Demoivre's theorem with positive integer power*
- *De Moivre's theorem with positive rational power*
- *The roots of complex number*
- *Representing the roots of the complex number in Argend's plane*

Key terms

- *De Moivre's thearem*
- *root*

Think and discuss

a If the complex number Z whose modulus r and argument θ, then find:

1) The modulus of the number Z^3

2) The argument of the number Z^3

b If Z is complex number and the principle amplitude of Z^3 is θ, then the principle amplitude of Z is ....

Learn

De moivre's theorem with positive integer power

If n is a positive integer number

Then $(\cos\theta + i\sin\theta)^n = \cos n\theta + , \sin n\theta$

Example

1 Express cos 3 θ in terms of cos θ

Solution

$\because (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ **(1) De Moivre's theorem.**

also $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + {}^3C_1\cos^2\theta\,(i\sin\theta).$

$+ {}^3C_2\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3$ **(binomial theorem).**

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ **(2)**

From (1) , (2) equating the real parts.

$\because \sin^2\theta + \cos^2\theta = 1$ $\quad\therefore \sin^2\theta = 1 - \cos^2\theta$

$\therefore \cos^3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta.$

Try to solve

1 Express sin 3 θ in terms of sin θ

De moivre's theorem with positive rational power

We know that $\cos\theta + i\sin\theta = \cos(\theta + 2r\pi) + i\sin(\theta + 2r\pi)$: where r is an integer.

If K is a positive number , then $(\cos\theta + i\sin\theta)^{\frac{1}{k}} = \cos\frac{\theta + 2r\pi}{k} + i\sin\frac{\theta + 2r\pi}{k}$

i.e $(\cos\theta + i\sin\theta)^{\frac{1}{k}}$ takes several values according to values of r, the number of this values equals K by putting $r = ..., -2, -1, 0, 1, 2 ....$ which makes the amplitude $\frac{\theta + 2r\pi}{k}$ included between $-\pi, \pi$

Example

2 Find in the trigonometric and exponential forms the roots of the equation $Z^4 = 8(1-\sqrt{3}\,i)$, then write the solution set.

Solution

$\because Z^4 = 8 - 8\sqrt{3}\,i \qquad \therefore x = 8, y = -8\sqrt{3}$

$r = \sqrt{(8)^2+(-8\sqrt{3})^2} = 16$, $\tan\theta = \frac{-8\sqrt{3}}{8} = -\sqrt{3}$

$\because x > 0 \;,\; y < 0 \qquad \therefore Z^4$ lies in the fourth quadrant

$\theta = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

$\therefore Z^4 = 16(\cos\frac{-\pi}{3}) + i\sin(\frac{-\pi}{3}))$

$\therefore Z = 2(\cos\frac{1}{4}(-\frac{\pi}{3} + 2\pi r) + i\sin\frac{1}{4}(-\frac{\pi}{3} + 2\pi r))$

when $r = 0$, **then** $Z_1 = 2(\cos(\frac{-\pi}{12}) + i\sin(\frac{-\pi}{12})) = 2e^{\frac{-\pi}{12}i}$

when $r = 1$, **then** $z_2 = 2(\cos(\frac{5\pi}{12}) + i\sin(\frac{5\pi}{12})) = 2e^{\frac{5\pi}{12}i}$

when $r = -1$, **then** $z_3 = 2(\cos(\frac{-7\pi}{12}) + i\sin(\frac{-7\pi}{12})) = 2e^{\frac{-7\pi}{12}i}$

when $r = 2$, **then** $z_4 = 2(\cos(\frac{11\pi}{12}) + i\sin(\frac{11\pi}{12})) = 2e^{\frac{11\pi}{12}i}$

$\therefore S.S. = \{2e^{\frac{-\pi}{12}i}, 2e^{\frac{5\pi}{12}i}, 2e^{\frac{-7\pi}{12}i}, 2e^{\frac{11\pi}{12}i}\}$

Try to solve

2 Find in C the solution set of $Z^4 = 2 + 2\sqrt{3}\,i$

Example

3 Find the roots of the equation $Z^3 = 1$, then represent the roots on Argend's plane.

Solution

$Z^3 = 1$

$= \cos 0^\circ + i\sin 0^\circ$

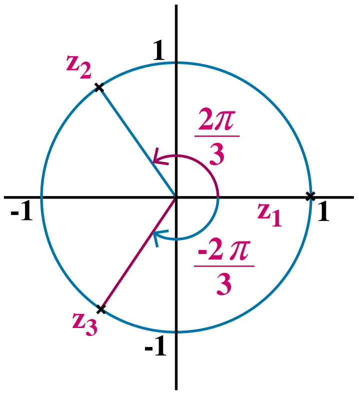
$\therefore Z = (\cos 0^\circ + i\sin 0^\circ)^{\frac{1}{3}}$

$= \cos \frac{1}{3}\ (2r\pi) + i \sin \frac{1}{3}\ (2r\pi)$

when $r = 0$, then $Z_1 = \cos 0^\circ + i \sin 0^\circ = 1$

when $r = 1$, then $Z_2 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$

when $r = -1$, then $Z_3 = \cos \frac{-2\pi}{3} + i \sin \frac{-2\pi}{3}$



We notice that the roots divide the circle whose center is the origin , and radius length is the unity into 3 equal arcs and the measure of each is 120°(the coordinates of the points form the vertices of an equilateral triangle).

Try to solve

3) Find the roots of the equation $Z^4 = 1$, then represent the roots on the Argand's plane.

The n^{th} roots

The equation $x^n = a$ where a is complex number has n roots in the form of $x = a^{\frac{1}{n}}$.
We can calculate them by finding the trigonometric form of a, then apply De Movre's theorem.
All roots lie on Argend's plane on a circle whose center is the origin and its radius length is $|a|^{\frac{1}{n}}$ to form a regular polygon of n vertices.

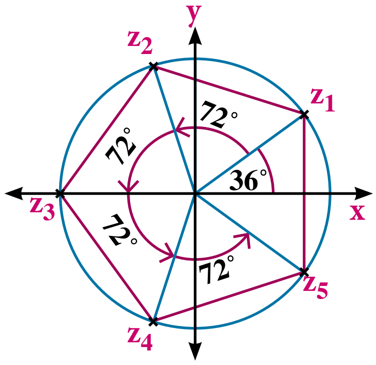
Example (The fifth roots of - 32)

4) Represent in Argand's diagram the fifth roots of - 32

Solution

The fifth roots of - 32 are the roots of the equation $Z^S = -32$

Convert the number - 32 into a trigonometric form.



$\therefore Z^S = 32 (\cos \pi + i \sin\pi)$

$\therefore Z = 32^{\frac{1}{5}} (\cos\pi + i \sin\pi)^{\frac{1}{5}}$

$= 2 (\cos \frac{1}{5} (\pi + 2r\pi) + i \sin \frac{1}{5} (\pi + 2r\pi))$.

The first root when $r = 0$

$\therefore Z = 2 (\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}) = 2 (\cos 36^\circ + i \sin 36^\circ)$

Thus, the measure of the angle between each two successive roots is $\frac{360}{5} = 72^\circ$

$Z_2 = 2 (\cos (36^\circ + 72^\circ) + i \sin (36^\circ + 72^\circ)) = 2 (\cos 108^\circ + i \sin 108^\circ)$

$Z_3 = 2 (\cos(36^\circ + 72^\circ) + i \sin (36^\circ + 2 \times 72^\circ)) = 2 (\cos 180^\circ + i \sin 180^\circ)$

$Z_4 = 2 (\cos (36^\circ + 3\times72^\circ) + i \sin (36^\circ + 3 \times 72^\circ)) = 2 (\cos 252^\circ + i \sin 252^\circ)$

$= 2 (\cos (-108^\circ) + i \sin (-108^\circ))$

$Z_5 = 2\,(\cos(36^\circ + 4 \times 72^\circ) + i \sin(36^\circ + 4\times72^\circ)) \quad = 2\,(\cos 324^\circ + i \sin 324^\circ)$

$= 2\,(\cos -36^\circ + i \sin -36^\circ)$

Try to solve

(4) Represent on Argand's diagram the sixth roots of 1

Example

(5) Find the square roots of $3+4i$

Solution

Let $(3+4i)^{\frac{1}{2}} = x + y\,i$ by squaring both sides

$\therefore\ 3 + 4i = x^2 - y^2 + 2xyi$

by equating the true part with the true part and the imaginary part with the imaginary part

$\therefore\ x^2 - y^2 = 3 \longrightarrow$ **(1)** , $\quad 2xy = 4 \longrightarrow$ **(2)** by squaring **(1)** , **(2)** and adding

$\therefore\ x^4 - 2x^2y^2 + y^4 + 4x^2y^2 = 9 + 16 \qquad \therefore\ x^4 + 2x^2y^2 + y^4 = 25$

$\therefore\ (x^2 + y^2)^2 = 25 \qquad \therefore\ (x^2 + y^2) = 5$ **(3)**

By adding **(1)**, **(3)** $\quad 2x^2 = 8 \quad \therefore \quad x = \pm 2$

when $x = 2$ by substituting in (2) $\quad y = 1$

when $x = -2 \qquad y = -1$

$\therefore$ the first root $= 2 + i \qquad \therefore$ the second root $= -2 - i$

Try to solve

(5) Find the square roots of the number $\frac{1}{\sqrt{2}}(1 + i)$ in the trigonometric form

Example

(6) Find in C the solution set of the equation $(1 - i)x^2 - (3 - i)x + 4 - 2i = 0$

Solution

The equation can be in the form of:

$$x^2 - \frac{6-4i}{1-i}x + \frac{9-7i}{1-i} = 0 \qquad x^2 - (5+i)x + 8 + i = 0$$

Use the general rule to solve the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{(5+i) \pm \sqrt{(5+i)^2 - 4 \times 1(8+i)}}{2}$$

$$x = \frac{(5+i) \pm \sqrt{24 + 10i - 32 - 4i}}{2}$$

$$x = \frac{(5+i) \pm \sqrt{-8+6i}}{2}$$

let $a + bi = \sqrt{-8+6i}$ **by squaring both sides**

$a^2 - b^2 + 2ab\,i = -8 + 6i$

$a^2 - b^2 = -8$ **(1)** $\qquad 2ab = 6$ **(2)**

$a^2 + b^2 = 10$ **(3)** $\qquad$ **from (1) , (3)** $a^2 = 1 \qquad \therefore a = \pm 1 \qquad \therefore b = \pm 3$

$\therefore a + b\,i = \pm(1 + 3i)$

$\therefore x = \dfrac{5 + i \pm (1 + 3i)}{2} \qquad x = 3 + 2i \quad$ or $\quad x = 2 - i$

Try to solve

6) Find in C the solution set of the equation $x^2 + (1 + i)x - 6 + 3i = 0$

Exercises 2 - 2

1) Use De moivre's theorem to prove each of the following identities:

a) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ $\qquad$ b) $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

2) Find in C the solution set in each of the following equations; write the roots in the form $x + y\,i$:

a) $Z^4 = 16$ $\qquad$ b) $Z^3 + 8 = 0$ $\qquad$ c) $Z^3 + 8i = 0$

3) Find the solution set of the equation $Z^5 + 243 = 0$ where $Z \in C$

4) Find the solution set of the equation of $Z^4 = 2 + 2\sqrt{3}\,i$ in the exponintial form .

5) Find the square roots for each of the following:

a) $2 - 2\sqrt{3}\,i$ $\qquad$ b) $1 - i$ $\qquad$ c) $8i$ $\qquad$ d) $3 + 4i$ $\qquad$ e) $5 - 12i$

6) Find the cubic roots of 8, then represent them on Argand's plane.

7) Find the fourth roots of - 1, then represent them on Argand's plane.

8) If $\dfrac{7 - 4i}{2 + i} = a + b\,i$, find the value of $(\sqrt{-b} + a\,i)^{\frac{3}{2}}$

9) Put the number $2\sqrt{2}\,(1 + i)$ in the trigonometric form, then find its square roots in the exponintial form.

10) If $Z = 8 - 6i$, find $Z^{\frac{3}{2}}$ in the algebraic form.

11) **Creative thinking :** prove that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos(2\theta) + 3)$

Cubic roots of unity

Unit two

2 - 3

You will learn

- The cubic roots of unity
- Properties of cubic roots of unity
- Representing the cubic roots of one geometrically
- Congugate of a $+\omega$, a i $+\omega$

Key terms

- Root
- Square root
- Cubic root
- Unit circle
- Conjugate

Materials

- Scientific calculator
- Graphical programs

Co-operation work :

Use De Moivre's theorem to find the solution set of the equation $Z^3 = 1$
Find the previous roots in algebraic form.
Find the sum of the 3 roots. What do you notice?

Learn

Cubic roots of unity

Use De Moivre's theorem we to find that the solution set of $Z^3 = 1$ is :

$$1 \,,\; -\frac{1}{2} + \frac{\sqrt{3}}{2}\, i \,,\; \frac{-1}{2} - \frac{\sqrt{3}}{2}\, i.$$

Also we notice that the square of one of the complex roots equals the other complex root, thus we can assume the cubic roots in the form of: $1, \omega, \omega^2$

where $\omega = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, $\omega^2 = -\frac{1}{2} \mp \frac{\sqrt{3}}{2} i$

Critical thinking :

Can you find the cubic roots of one using the algebraic form of the complex number ?

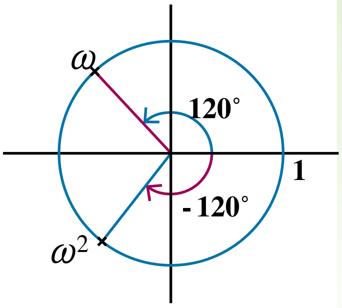
Properties of cubic roots of one:

If 1 , ω , ω^2 are the cubic roots of one, then

1- $1 + \omega + \omega^2 = 0$ (the sum of the roots = 0)
$(1 + \omega = -\omega^2 ,\; 1 + \omega^2 = -\omega ,\; \omega + \omega^2 = -1)$

2- $\omega^3 = 1$
$\left(\frac{1}{\omega} = \omega^2 ,\; \frac{1}{\omega^2} = \omega\right)$

3- The cubic roots of one lies on a circle whose center is the origin, it's radius length is 1 and form vertices of an equilateral triangle.



4- $(\omega - \omega^2) = \pm\sqrt{3}\; i$,

Example

1 If 1 , ω , ω^2 , are the cubic roots of one, find the value of :

a $5 + 5\omega + 5\omega^2$

b $(1 - \frac{2}{\omega} - \frac{2}{\omega^2})(3 + 5\omega + 5\omega^2)$

Solution

a The expression = $5(1 + \omega + \omega^2)$ by taking 5 a common factor

$= 5 \times 0 = 0$

b The expression = $(1 - \frac{2}{\omega} - \frac{2}{\omega^2})(3 + 5\omega + 5\omega^2)$ by substituting $\frac{1}{\omega} = \omega^2, \frac{1}{\omega^2} = \omega$

$= (1 - 2\omega^2 - 2\omega)(3 + 5\omega + 5\omega^2) = (1 - 2(\omega^2 + \omega))(3 + 5(\omega + \omega^2))$

$= (1 - 2(-1))(3 + 5(-1)) = (1 + 2)(3 - 5) = -6$

Try to solve

1 If 1 , ω , ω^2 are the cubic roots of one, find the value of:

a $(2 + 5\omega + 2\omega^2)^5$

b $(\omega + \frac{1}{\omega})^2 (\omega^2 + \frac{1}{\omega^2})^3$

Example

2 Prove that $[\frac{5 - 3\omega^2}{5\omega - 3} - \frac{2 - 7\omega}{2\omega^2 - 7}]^4 = 9$

Solution

The expression = $[\frac{5 - 3\omega^2}{5\omega - 3} - \frac{2 - 7\omega}{2\omega^2 - 7}]^4 = [\frac{5\omega^3 - 3\omega^2}{5\omega - 3} - \frac{2\omega^3 - 7\omega}{2\omega^2 - 7}]^4$

$= [\frac{\omega^2(5\omega - 3)}{5\omega - 3} - \frac{\omega(2\omega^2 - 7)}{2\omega^2 - 7}]^4 = [\omega^2 - \omega]^4 = [\pm\sqrt{3}\, i]^4 = 9$

Try to solve

2 Prove that = $[\frac{a + \omega b + \omega^2 c}{\omega^2 a + b + c\omega} - \frac{c + \omega^2 - b}{\omega c + b}]^8 = 81$

Example

3 Prove that $x = \frac{-1 + \sqrt{-3}}{2}$ is one of the roots of the equation $x^{10} + x^5 + 1 = 0$

Solution

$\because x = \frac{-1 + \sqrt{-3}}{2} = \frac{-1 + \sqrt{3}\, i}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

i.e. x represents one of the complex roots of one

when $x = \omega$, then $\omega^{10} + \omega^5 + 1 = \omega^1 + \omega^2 + 1 = 0$

when $x = x^2$ then $\omega^{20} + \omega^{10} + 1 = \omega^2 + \omega + 1 = 0$

Try to solve

3. Form the quadratic equation whose roots are $(1 + \omega - \omega^2)^3$, $(1 - \omega + \omega^2)^3$

Exercises 2 - 3

If 1 ,ω ,ω^2 are the cubic roots of one :

Complete the following:

1. $(2 + 5\omega + 2\omega^2)^2 =$ ....

2. $(\omega - \omega^2)^4 =$ ....

3. $\left(\omega + \frac{1}{\omega}\right)^2 \left(\omega^2 + \frac{1}{\omega^2}\right)^2 =$ ....

4. If $x = \frac{-1 + \sqrt{3}\, i}{2}$, then $x^8 + x^4 =$ ....

5. $\left(\frac{1}{\omega + 2}\right)\left(1 + \omega - \frac{3}{\omega}\right) =$ ....

6. $1 + 3\omega + 3\omega^2 =$ ....

7. If $a = 2\omega - 3\omega^2$, $b = 3 + 5\omega^2$,then $a^2 + b^2 =$ ....

8. $\sum_{r=1}^{5} \omega^r =$ ....

Choose the correct Answer from the given:

9. The conjugate of ω is ....

a) ω b) ω^2 c) 1 d) $-\omega$

10. $\left(\omega^2 + \frac{1}{\omega}\right)\left(1 + \frac{1}{\omega^2}\right)^2 =$ ....

a) 2 b) 0 c) - 3 d) - 5

11. $(a + b\omega + a\omega^2)(a + b\omega^2 + a\omega^4) =$ ....

a) 1 b) $a - b$ c) $(a - b)^2$ d) $b^2 - a^2$

12. $\left(1 + 2\omega^5 + \frac{1}{\omega^2}\right)\left(1 + 2\omega + \frac{1}{\omega^4}\right) =$ ....

a) 0 b) 1 c) - 1 d) 2

13. $\frac{a - d\omega}{a\omega^2 - d} - \omega^2 =$ ....

a) 3i b) $\pm\sqrt{3}\, i$ c) - 3 d) 3

14. If $(1 + \omega)^7 = a + b\omega$ where a and b are real numbers, then (a , b) = ...

a) (0 , - 1) b) (1 , 1) c) (0 , 1) d) (1 , - 1)

15. If $(1 + \omega^2)^n = (1 + \omega)^n$, then the least value of the positive integer n is ....

a) 2 b) 3 c) 5 d) 6

16. $1 + \omega + \omega^2 + \omega^3 + \dots + \omega^{100} =$ ....

a) 0 b) 1 c) ω d) $-\omega^2$

17. If $Z = \omega^x$ then $|Z| =$ .... where x is a positive integer

a) 1 b) ω c) x d) ω^2

18. $\sum_{r=1}^{6} 1 + \omega^r =$ ....

a) 0 b) 6 c) 1 d) $1 + \omega$

19. Prove each of the following identities:

a) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)(1 - \omega^8 + \omega^{16}) = {}^4 2$

b) $\left(\frac{\omega}{1 + 2\omega}\right)^2 + \left(\frac{1 + 2\omega^2}{\omega^2}\right)^2 = \frac{-10}{3}\omega^2$

c) $\left[\frac{1}{1 + \omega i} - \frac{\omega + i}{1 + \omega^2 i}\right]^8 = 16$

d) $\left[\frac{5 - 3\omega^2}{5\omega - 3} - \frac{2 - 7\omega}{2\omega^2 - 7}\right]^2 = -3$

e) $(1 + \omega^2)^8 = \omega^2$

f) $(1 - \omega^2 + \omega^4)^2 + (1 + \omega^2 + \omega^4)^2 = 4\omega$

20. Find value of each of the following:

a) $5 + 3\omega + 3\omega^2$

b) $(1 + \omega + 2\omega^2)^2 + (1 + 2\omega + \omega^2)^2$

c) $\frac{\omega^2(\omega - 1)(\omega^2 - 1)}{(2\omega + 1)(\omega^2 + 2)}$

d) $\left[\frac{1}{1 + 3\omega^2} - \frac{1}{1 + 3\omega}\right]^2$

e) $(1 + \frac{1}{\omega} + i)(1 + \frac{1}{\omega^2} + i)$

21. If $x = \frac{-1 + \sqrt{3}\, i}{2}$, prove that $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x = 0$.

22. If $\frac{1}{1 + \omega}$, $\frac{1}{1 + \omega^2}$ are the two roots of a quadratic equation, find the equation .

23. If $Z = 2(\omega + i)(\omega^2 + i)$, find all different forms of the number Z. and the square roots of Z in the trigonometric form.

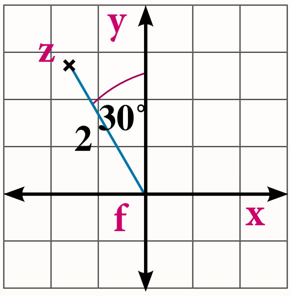
24. **Creative thinking:** Find value of n which makes $(2 + 5\omega + 2\omega^2)^n = (2 + 2\omega + 5\omega^2)^n$:

25. Find : a) $\sum_{= \text{zero}}^{10} \omega^r$ b) $\sum_{= \text{zero}}^{10} (1 + \omega^r + \omega^{2r})$

General Exercises

Complete the following :

1. If $Z = \frac{2 + i}{2 - i}$, then $|Z| =$ ....



2. The trigonometric form of the number Z represented in the opposite figure in the form of Argand's diagram is ....

3. If $Z = \sin\theta - i\cos\theta$, then arg (Z)= ....

4. The conjugate of the number $i + \omega^2$ is ....

5. $1 + 2\omega + 2\omega^2 =$ ....

6. If Z_1, Z_2 , ... Z_6 represent the sixth roots of unity on Argand's plane, then $(Z_r \ \hat{f} \ Z_{r+1}) =$ .... where $1 \leqslant r \leqslant 5$

Choose the correct answer from the following:

7. If arg Z_1 is θ_1, Arg Z_2 is θ_2 , then Arg Z_1Z_2 is:

   a $\theta_1 + \theta_2$ b $\theta_1 \times \theta_2$ c $\theta_1 - \theta_2$ d $\pi - (\theta_1 + \theta_2)$

8. Which of the following represents the algebraic form of $2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$:

   a $-\sqrt{3} + i$ b $-1 + \sqrt{3}\, i$ c $-\sqrt{2} + \sqrt{2}\, i$ d $-\sqrt{3} - \sqrt{3}\, i$

9. If A $(\sqrt{3}\ , -1)$ represents a complex number in Argend's plane, then the modulus and the argument of Z are ....

   a $(2, \frac{\pi}{6})$ b $(2, \frac{5\pi}{6})$ c $(2, -\frac{5\pi}{6})$ d $(2, \frac{-\pi}{6})$

10. The real part of the complex number whose modulus $\sqrt{2}$ and argument $\frac{7\pi}{6}$ is ....

   a $-\sqrt{3}$ b $-\frac{\sqrt{6}}{2}$ c $-\frac{\sqrt{2}}{6}$ d $\frac{\sqrt{6}}{3}$

11. The conjugate of the number $1 + \omega$ is

   a $1 - \omega$ b $-\omega^2$ c $1 + \omega^2$ d $1 - \omega^2$

12. The fifth roots of unity are represented on Argand's plane by the vertices of ....

   a Equilateral triangle   b square   c Reguler pentagon   d regulare hexagon

13 If a is real number, then the conjugate of $\frac{a^3 + i}{a^2 + ai - 1}$ is ....

a $a - i$ b $a + i$ c $\frac{a^3 - i}{a^2 - ai - 1}$ d $\frac{a^3 + i}{a^2 - ai - 1}$

14 If $Z = (\frac{1}{2} + \frac{\sqrt{3}}{2} i)^n$ where n is a positive integer and $Z = 1$, then the least value of n is = ...

a 9 b 6 c 3 d 1

15 If $|Z| = |Z - 2|$, then the real part of $Z =$ ....

a 1 b - 1 c - 2 d 2

16 $e^{\theta i} + e^{-\theta i} =$ ....

a $e^{2\theta i}$ b $2 \cos \theta$ c $2 \sin \theta$ d $e^{-2\theta i}$

17 If $|Z| = 10$, then $Z \bar{Z} =$ ....

a 10 b 100 c 1 d - 100

18 Express each of the following in the form of $x + y i$:

a $(\cos \frac{3\pi}{11} + i \sin \frac{3\pi}{11})(\cos \frac{8\pi}{11} + i \sin \frac{8\pi}{11})$

b $3(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}) \times \sqrt{2} (\cos \frac{\pi}{3} - i \sin \frac{\pi}{3})$

c $\dfrac{\sqrt{2}(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})}{\frac{1}{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})}$

19 If Z_1 and Z_2 are two complex numbers where $Z_1 = -9 + 3\sqrt{3}\, i$, $|Z_2| = \sqrt{3}$, $\arg Z_2 = \frac{7\pi}{12}$, find each of the following complex number in the form of $r(\cos\theta + i \sin\theta)$ where $-\pi < \theta \leqslant \pi$

a Z_1 b Z_2 c $Z_1 Z_2$ d $\frac{Z_1}{Z_2}$

20 Express each of the following in the exponential form:

a $Z_1 = 7$ b $Z_2 = -5i$ c $Z_3 = \frac{1}{2}(\cos 60° + i \sin 60°)$

d $Z_4 = \frac{7}{2 + \sqrt{3}\, i}$ e $Z_5 = -4(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6})$

21 If $\theta \in]-\pi, \pi]$, find modulus and argument of the number $Z = 1 + \cos\theta + i \sin\theta$

22 If $Z = \frac{(1 + i)(2 - i)}{(1 - i)(3 + i)}$, find $|Z|$

23 **Creative thinking :** Use the complex numbers to prove the following relation :

$$\tan^{-1}(\sqrt{3}) + \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{2}$$

Unit Summary

- **Complex number** for each $x , y \in R$ thus $Z = x + y\,i$ is called complex number whose real part is x and the imaginary part is y where $i^2 = -1$

- **The conjugate of the complex number**: if $Z = x + y i$, then it's conjugate is $\overline{z} = x - y i$ and $Z + \overline{z}$ = real number, $Z\,\overline{z}$ = real number

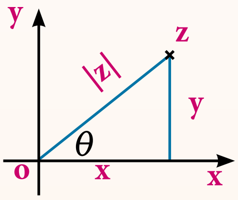
- **Properties of the conjugate**:

  1) $(\overline{Z_1 + z_1}) = \overline{z_1} + \overline{z_2}$ **2)** $(\overline{z_1\, z_1}) = (\overline{z_1})(\overline{z_2})$ **3)** $\left(\overline{\dfrac{z_1}{z_2}}\right) = \dfrac{\overline{z_1}}{\overline{z_2}}$

- **Geometrical representation of complex number**: The complex number $Z = x + y\,i$ is represented by point (x , y) in Argand's plane .

- **The modulus and the amplitude of the complex number**: If point (x , y) represents the complex number Z on Argand's plane, then $|Z| = \sqrt{x^2 + y^2}$ amplitude of Z is got from $\cos\theta = \dfrac{x}{r}$, $\sin\theta = \dfrac{y}{r}$

- **Properties of modulus and amplitude of a complex number**



  1) $|Z| = |\overline{z}|$ **2)** $Z\,\overline{z} = |Z|^2$

  3) $|Z_1 Z_2| = |Z_1||Z_2|$ **4)** $\left|\dfrac{Z_1}{Z_2}\right| = \dfrac{|Z_1|}{|Z_2|}$

  5) $|Z_1 + Z_2| \leqslant |Z_1| + |Z_2|$

  6) The amplitude of a complex number can take an infinite number of values that each differ by amplitude of 2π.

  7) The amplitude which belongs to the interval $]-\pi , \pi]$ is called the principle amplitude of a complex number .

  8) $\arg \overline{z} = -\arg Z$ **9)** $\arg(-Z) = -\pi + \arg Z$

  10) $\arg\dfrac{1}{z} = -\arg Z$

- **The trigonometric form of a complex number**: $Z = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and θ is the principle amplitude

- **Multiplying and dividing complex numbers in a trigonometric form**:

  If $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then:

  $Z_1 Z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

  $\dfrac{Z_1}{Z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

The exponential form of the complex number (Euler's form) if Z is a complex number whose modulus is r and principle amplitude is θ, then:

$Z = r e^{\theta i}$ where θ in radian measure.

$e^{\theta i} = \cos\theta + i \sin\theta$, $e^{-\theta i} = \cos\theta - i \sin\theta$

ruler expansion for sin x , cos x , e^x functions

$$\sin x = \frac{x}{\lfloor 1} - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \dots + (-1)^{n-1} \times \frac{x^{2n+1}}{\lfloor 2n+1} + \dots$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + (-1)^{n-1} \times \frac{x^{2n-2}}{\lfloor 2n-2}$$

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots + \frac{x^{n-1}}{\lfloor n-1} + \dots$$

De Moivre's theorem: If n is a positive real number, then:

1) $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$

2) If K is (+ve) number , then $(\cos\theta + i \sin\theta)^{\frac{1}{k}} = \cos\frac{\theta + 2r\pi}{K} + i \sin\frac{\theta + 2r\pi}{K}$

thus $(\cos\theta + i \sin\theta)^{\frac{1}{k}}$ takes different values according to r and the number of these different values equals K values which we get by putting r = ... , - 2 , - 1 , 0 , 1 , 2 .... that makes the amplitude $\frac{\theta + 2r\pi}{K}$ included between $-\pi, \pi$

The cubic roots of one: if $Z^3 = 1$ then $Z = 1 , -\frac{1}{2} + \frac{\sqrt{3}}{2} i , -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

and these roots can be denoted by $1, \omega, \omega^2$

where $\omega = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, $\omega^2 = -\frac{1}{2} \mp \frac{\sqrt{3}}{2} i$

Properties of the cubic roots of one:

1) $\omega^3 = 1$ **2)** $1 + \omega + \omega^2 = 0$ **3)** $\omega^2 - \omega = \pm\sqrt{3}\, i$

The n^{th} roots of one: if $Z^n = 1$,

Then $Z = (\cos 0^\circ + i \sin 0^\circ)^{\frac{1}{n}} = \cos\frac{2\pi k}{n} + i \sin\frac{2\pi k}{n}$ where $k \in \mathbf{y}$, $\frac{2\pi k}{n} \in]-\pi, \pi]$

The n^{th} roots of one is represented in the Argand's plane by a regular polygon with n vertices which lie on a circle whose center is origin point and radius length equals 1.

For more activities and exercises , visit www.sec3mathematics.com.eg

Accumulative test

1. Determine the quadrant at which θ lies in each of the following :

   a) $\cos\theta > 0$, $\sin\theta > 0$ b) $\cos\theta < 0$, $\sin\theta < 0$

   c) $\cos\theta > 0$, $\sin\theta > 0$

2. Find the sum and product of the two roots of the equation $x^2 - 3x + 1 = 0$

3. Find modulus and the amplitude for each of the following :

   a) $Z = \sqrt{3} + i$ b) $Z_2 = -1 + i$ c) $Z_3 = 4i$

   d) $Z_4 = -5$ e) $Z_5 = 3 - 4i$

4. Find in the simplest form $Z = \frac{1 + 4i + i^2}{2 - i - 2i^2 - i^3}$, then find the square roots of Z in a trigonometric form.

5. If Z is a complex number, find the solution set of the equation $2Z - 3\overline{z} = 5 + 10i$

6. If $Z = 4 + 4\sqrt{3}\,i$, find the exponential form of Z , the cubic roots of Z and represent these roots on Argand's diagram .

7. Find all the forms of the number $Z = \frac{-\sqrt{3} - i}{\sqrt{3} - i}$, the square roots of Z and represent these roots on Argand's plane .

8. If $Z_1 = \sqrt{3}\,(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})$, $Z_2 = 2\,(\sin 30^\circ + i\sin 30^\circ)$,

   find $Z_1 Z_2$ in the exponential form and the trigonometric form of Z where $Z = (Z_1 Z_2)^{\frac{1}{4}}$

9. Put $Z = \frac{-4}{1-\sqrt{3}\,i}$ in the exponential and trigonometric forms, then :

   1) Prove that Z^6 is real number. **2)** prove that $\frac{1}{Z} = \frac{1}{2} e^{\frac{2\pi}{3} i}$

10. If $Z = e^{\theta i}$, find modulus and principle amplitude of the number $\frac{1 + Z}{1 - Z}$

11. If $Z = \frac{-1 + \sqrt{3}\,i}{2}$ if $Z_1 = \frac{1 + Z}{1 - Z}$, find Z_1 and its square roots in the trigonometric form.

12. Prove that :

   a) $(1 - \frac{2}{\omega^2} + \omega^2)(1 + \omega - \frac{5}{\omega}) = 18$

   b) $\frac{3 + 5\omega + 3\omega^2}{1 - 2\omega - 4\omega^2} + \frac{3 + 5\omega^2 + 3\omega}{1 - 2\omega^2 - 4\omega} = \frac{-2}{19}$

First: Algebra

Unit three

Determinants and Matrices

Unit introduction

Matrices are much important materials used in all the branches of mathematics. they are much necessary for the linear algebra. Matrices are a mathematical concept which do a great role in all branches of knowledge. Ancient chinese had been the first to discover the matrices. The scientist Kelly (1821-1895) used the matrices in a regular way. He stated a system in the form of columns and rows. At the present time, matrics are used by the specialists in economics, sociology and psychology furthermore matrices play a great role in mathematics and other applications in physics, chemistry and other applied science. On the other hand, the determineants are representing arthimtic values for square matrices. The Japanese scientist Sekikowa was the first to use them to solve linear equations in 1683. Later, this science has been developed by other scientists to solve the linear equations and some other applications in different sciences.

Unit objectives

By the end of this unit and doing all the activities included, the student should be able to:

- Deduce the properties of the determinants
- Solve different problems using the properties of the determinants.
- Identify the multiplicative inverse of the matrix of the order 3×3 using the cofactors of the matrix.
- Solve a system of linear equations using the multiplicative inverse of the matrix.
- Recognize the Homogeneous and non-homogeneous linear equations.
- Identify the rank of the matrix of coefficients and the ranke of the augmented matrix.
- Deduce the relation between the ranks of the matrix of coefficients and the augmented matrix and the possiblilities of solutions.

Key terms

- *Determinants*
- *Second – order determinant*
- *Third – order determinant*
- *Row*
- *column*
- *Motrix*
- *Element*
- *Rank*
- *Row matrix*
- *column natrix*
- *Square matrix*
- *Zero matrices*
- *Equal matrices*
- *Rank of a matrix*
- *Augmented Matrix*
- *Homogenuous equation*
- *Non Homgenuous equation*
- *Adjoint matrix*
- *co factor matrix*
- *Linear Equation*

Materials

- *Scientific calculator*

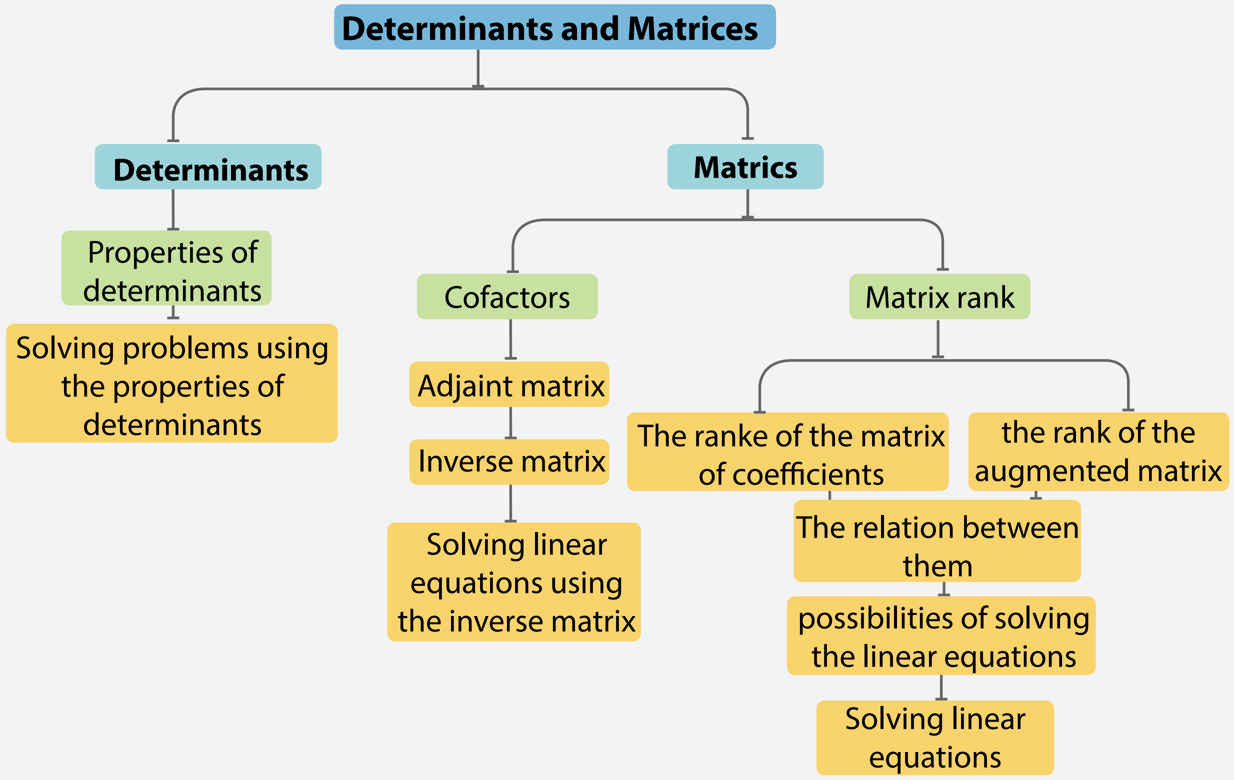
Unit Lessons

Lesson (3 – 1): Determinants.

Lesson (3 – 2): matrices.

Lesson (3 – 3): Solving linear equations using the multiplicative inverse of the matrix

Unit chart



Unit three

3 - 1 Determinants

You will learn

- Properties of determinants.
- Solving miscellaneous problems using properties of determinants .

Key terms

- Determinants
- Second - degree determinant
- Third - degree determinant
- Row
- column
- Main diagonal
- Triangular form

Materials

- Scientific calculator

Introduction

You have studied matrices and determinants and you known that each square matrix has a determinant and the determinant 2×2 is named by second-order determinant and the determinant 3×3 is named by third-order determinant and son on ...

You have also known how to find the value of the determinant, for **example:** $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

The value of $\begin{vmatrix} 3 & 4 \\ -5 & 6 \end{vmatrix} = 3 \times 6 - 4\,(-5) = 38$

Also you have learned how to find the value of the determinant $\begin{vmatrix} a & d & x \\ b & e & y \\ c & f & z \end{vmatrix}$

By using the cofactor method, if we denote to the value of the determinant by $\triangle$

then $\triangle = a \begin{vmatrix} e & y \\ f & z \end{vmatrix} - d \begin{vmatrix} b & y \\ c & z \end{vmatrix} + x \begin{vmatrix} b & e \\ c & f \end{vmatrix}$ **by using the elements of first row**

Think and discuss

1 If $\triangle_1 = \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$, then $\triangle_1 =$ ....

2 If $\triangle_2 = \begin{vmatrix} 3 & 2 \\ -1 & 5 \end{vmatrix}$, then $\triangle_2 =$ ....

3 What is the relation between $\triangle_1$ and $\triangle_2$? Are they equal ? Explain.

4 What is the relation between the rows of $\triangle_1$ and the columns of $\triangle_2$? what can you deduce?

Learn

Main properties of determinantes

Property (1)

In any determinant if the rows replace the columns in the same order, the value of the determinant is unchanged

$$\triangle = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ b_{12} & b_{22} & b_{32} \\ c_{13} & c_{23} & c_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{vmatrix}$$ this can be proved by expanding the two determinants.

Example

1 Prove that $\begin{vmatrix} 2 & -3 & -1 \\ 1 & 0 & 4 \\ -2 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -2 \\ -3 & 0 & 5 \\ -1 & 4 & 2 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & -3 & -1 \\ 1 & 0 & 4 \\ -2 & 5 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 4 \\ 5 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} = 2\,(0 - 20) + 3(2 + 8) - (5 - 0) = -15$$

$$\begin{vmatrix} 2 & 1 & -2 \\ -3 & 0 & 5 \\ -1 & 4 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 5 \\ 4 & 2 \end{vmatrix} - \begin{vmatrix} -3 & 5 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} -3 & 0 \\ -1 & 4 \end{vmatrix} = 2\,(0 - 20) - (-6 + 5) - 2(-12, -0) = -15$$

So : $\begin{vmatrix} 2 & -3 & -1 \\ 1 & 0 & 4 \\ -2 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -2 \\ -3 & 0 & 5 \\ -1 & 4 & 2 \end{vmatrix}$

Try to solve

1 Prove that $\begin{vmatrix} 1 & 2 & -3 \\ 5 & 4 & 1 \\ -1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -1 \\ 2 & 4 & 3 \\ -3 & 1 & 2 \end{vmatrix}$

Property (2)

The value of the determinant is unchanged by evaluating it in terms of the elements of any of its rows (columns)

Example

2 Find the value of $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 4 & 3 \\ 5 & -1 & 0 \end{vmatrix}$ using the elements of the first column once and the elements of the first row once more.

Solution

First : By using the elements of first column

$$\triangle = \begin{vmatrix} 4 & 3 \\ -1 & 0 \end{vmatrix} + 5 \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = (0 + 3) + 5(6 + 4) = 53$$

Second: By using the elements of first row

$$\triangle = \begin{vmatrix} 4 & 3 \\ -1 & 0 \end{vmatrix} - 2 \begin{vmatrix} 0 & 3 \\ 5 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = (0 + 3) - 2(0 - 15) - (0 - 20) = 3 + 30 + 20 = 53$$

Try to solve

(2) Find the value of the determinant $\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix}$ using the elements of the first row once and the elements of the second column once more.

Property (3)

The value of the determinant vanishes in each of the following two cases

First: if all the elements of any row (column) in any determinant equal zero, then the value of the determinant = 0

The value of the determinant $\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ \text{zero} & \text{zero} & \text{zero} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = 0$

This can be proved by expanding the determinant using the elements of second row, then: $\triangle = 0$

Second: If the corresponding elements of any two rows (columns) in any determinant are equal, then the value of the determinant = zero

i.e. $\begin{vmatrix} c & b & c \\ c & b & c \\ x & y & z \end{vmatrix} = 0$

Since the corresponding elements of the first and second rows are equal (prove that).

Example

(3) Without expanding the determinant, prove that $\begin{vmatrix} 1 & -5 & 8 \\ 3 & 7 & 9 \\ 1 & -5 & 8 \end{vmatrix} = 0$

Solution

In the determinant, we find $R_3 = R_1$ $\therefore$ The value of the determinant = 0

Try to solve

3) Without expanding the determinant, prove that $\begin{vmatrix} 3 & -1 & 3 \\ 2 & 5 & 2 \\ -1 & 7 & -1 \end{vmatrix} = 0$

Property (4)

If there is a common factor in all the elements of any row (column) in a determinant, then this factor can be taken outside the determinant

Example

4) Without expanding the determinant, find the value of $\begin{vmatrix} 3 & 2 & -7 \\ 4 & 6 & 2 \\ 10 & 15 & 5 \end{vmatrix}$

Solution

Take 2 as a common factor of R_2 and 5 as a common factor of R_3

$$\begin{vmatrix} 3 & 2 & -7 \\ 4 & 6 & 2 \\ 10 & 15 & 5 \end{vmatrix} = 2 \times 5 \begin{vmatrix} 3 & 2 & -7 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 10 \times 0 = 0$$

Since the corresponding elements in R_2 and R_3 are equal"Try to prove that by another method".

Try to solve

4) If $\begin{vmatrix} a & d & m \\ b & e & n \\ c & f & z \end{vmatrix} = 10$ find the value of $\begin{vmatrix} 2a & 2d & 2m \\ b & e & n \\ -4c & -4f & -4z \end{vmatrix}$

Property (5)

In any determinant, if the positions of two rows (columns) are interchanged. The value of the resulting determinant equals the additive inverse of the value of the original determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = - \begin{vmatrix} a & b & c \\ x & y & z \\ d & e & f \end{vmatrix}$$

by interchanging the second and the third rows

Example

5) Without expanding the determinant, prove that: $\begin{vmatrix} 2 & -1 & 2 \\ 3 & 3 & 4 \\ 5 & 7 & 6 \end{vmatrix} + \begin{vmatrix} 3 & 3 & 4 \\ 2 & -1 & 2 \\ 5 & 7 & 6 \end{vmatrix} = 0$

Solution

By interchanging the first and the second rows in the first determinant

$$\therefore -\begin{vmatrix} 3 & 3 & 4 \\ 2 & -1 & 2 \\ 5 & 7 & 6 \end{vmatrix} + \begin{vmatrix} 3 & 3 & 4 \\ 2 & -1 & 2 \\ 5 & 7 & 6 \end{vmatrix} = -\triangle + \triangle = \text{zero}$$

Property (6)

In any determinant, if all the elements of any row (column) are written as a sum of two elements, then the value of the determinant can be written as a sum of two determinants

$$\begin{vmatrix} c_{11}+m & a_{12} & a_{31} \\ c_{21}+n & a_{22} & a_{23} \\ c_{31}+h & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} m & c_{12} & c_{13} \\ n & c_{22} & c_{23} \\ h & c_{32} & c_{33} \end{vmatrix}$$

This can be proved by expanding the determinants.

Example

6 Without expanding the determinant, prove that: $\begin{vmatrix} l & m & n \\ -x & y & z \\ f & k & r \end{vmatrix} + \begin{vmatrix} -l & m & n \\ x & y & z \\ -f & k & r \end{vmatrix} = 0$

Solution

We can write a determinant whose value equals the sum of the values of two determinants in the left side.

(notice that the two determinants of the left side have the same two columns C_2, C_3 :

$$\text{Side} = \begin{vmatrix} l-l & m & n \\ -x+x & y & z \\ f-f & k & r \end{vmatrix} = \begin{vmatrix} 0 & m & n \\ 0 & y & z \\ 0 & k & r \end{vmatrix} = 0$$

(Since all the elements of C_1 are zeros)

Critical thinking: Can you use other methods to find the value of the determinants without expanding them? Mention one of these methods.

Try to solve

5 Find the determinant $D = D_1 + D_2 + D_3$ where

$$D_1 = \begin{vmatrix} 2 & 3 & 4 \\ 5 & -2 & 3 \\ 0 & 4 & -1 \end{vmatrix}, \quad D_2 = \begin{vmatrix} 2 & 3 & 4 \\ -1 & 5 & 2 \\ 0 & 4 & -1 \end{vmatrix}, \quad D_3 = \begin{vmatrix} 2 & -1 & 4 \\ 4 & 1 & 5 \\ 0 & 1 & -1 \end{vmatrix}$$

Property (7)

If we add all the elements of any row (column) to the multiples of the elements of another row (column), the value of the determinant is unchanged.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + k\,b_2 & c_1 + k\,c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We add to the elements of the first row the elements of the second row multiplied by K and this operation is denoted by $R_1 + KR_2$. We can prove that by parting the elements of the first row in the right determinant according to the previous property as a sum of two determinants one of them is the left determinant and the other its value is equal to Zero

Example

7 Without expanding the determinant, find the value of: $\begin{vmatrix} 3 & 8 & 12 \\ 9 & 21 & 27 \\ 20 & 44 & 52 \end{vmatrix}$

Solution

By Multiplying the elements of the first column by -2 and adding to the corresponding elements of the second row.

i.e: $-2C_1 + C_2$ **and:** $-2C_1 + C_3$

$$\begin{vmatrix} 3 & 8 & 12 \\ 9 & 21 & 27 \\ 20 & 44 & 52 \end{vmatrix} = \begin{vmatrix} 3 & 8 - 2\times 3 & 12 - 2\times 3 \\ 9 & 21 - 2\times 9 & 27 - 2\times 9 \\ 20 & 44 - 2\times 20 & 52 - 2\times 20 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 6 \\ 9 & 3 & 9 \\ 20 & 4 & 12 \end{vmatrix}$$

By multiplying the elements of second column by -3 and adding to the corresponding elements of the third row

$\therefore$ The value of the determinant = 0

Try to solve

6 Without expanding the determinant, find the value of $\begin{vmatrix} 2 & 5 & -7 \\ -3 & -1 & 4 \\ 3 & 1 & -4 \end{vmatrix}$

Property (8)

If we multiply the elements of any row (column) by the cofactors of the corresponding elements in another row (column) and we sum the results, then the sum is Zero

In the determinant $\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$

Since the elements of the first row are a_{11} , a_{12} , a_{13}, the coefactors corresponding to the elements of the second row are:

$$(-1)^{2+1} \times M_{12}\ ,\quad (-1)^{2+2} \times M_{22}\ ,\quad (-1)^{2+3} \times M_{32}$$

$$\text{Determinant} = -a_{11}\begin{vmatrix} a_{21} & a_{31} \\ a_{23} & a_{33} \end{vmatrix} + a_{21}\begin{vmatrix} a_{11} & a_{31} \\ a_{13} & a_{33} \end{vmatrix} - a_{31}\begin{vmatrix} a_{11} & a_{21} \\ a_{13} & a_{23} \end{vmatrix}$$

$$= -\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{11} & a_{21} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \text{zero} \quad \textbf{(Since: } R_1 = R_2\textbf{)}$$

For example: if $A = \begin{vmatrix} 2 & 4 & 7 \\ 3 & -2 & 1 \\ 1 & 5 & 6 \end{vmatrix}$ then the elements of R_1 are 2, 4, 7

the cofactors corresponding to the third row are:

$$(-1)^{3+1}\begin{vmatrix} 4 & 7 \\ -2 & 1 \end{vmatrix}\ ,\quad (-1)^{3+2}\begin{vmatrix} 2 & 7 \\ 3 & 1 \end{vmatrix}\ ,\quad (-1)^{3+3}\begin{vmatrix} 2 & 4 \\ 3 & -2 \end{vmatrix}$$

i.e.: 1(4 + 14) , -1 (2 - 21) , 1 (-4 - 12)

i.e.: 18 , 19 , -16

Summution of the products of elements of R_1 by the cofactors of the elements of R_3

$$= 2 \times 18 + 4 \times 19 + 7 \times -16 = \text{zero.}$$

Determinant of a triangular form

If the determinant is written as

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & 0 \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ 0 & a_{22} & c_{32} \\ 0 & 0 & a_{33} \end{vmatrix}$$

Main diagonal ↗ Main diagonal ↗

This form is called lower triangular form and upper triangular form respectively.

All the elements of the determinant lying above the main diagonal as in first case or below the main diagonal as in second case are zeros and the elements of a_{11} , a_{22} , a_{33} are called the elements of the main diagonal.

Property (9)

The value of the determinant in the triangular form equals the product of the elements of the main diagonal.
The value of the determinant of the previous form = $a_{11} \times a_{22} \times a_{33}$

Example

8 Without expanding the determinant , prove that: $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$

Solution

Multiplying the elements of the first row by -1 and adding them to the corresponding elements of the second and the third rows

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & b - a & b^2 - a^2 \\ 0 & c - a & c^2 - a^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b - a & (b - a)(b + a) \\ 0 & c - a & (c - a)(c + a) \end{vmatrix}$$

Taking (b - a) common from second row and taking (c - a) common from third row

$$= (b - a)(c - a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b + a \\ 0 & 1 & c + a \end{vmatrix}$$

Multiplying the elements of the second row by -1 and adding them to the corresponding elements of the third row

$$= (b - a)(c - a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b + a \\ 0 & 0 & c - b \end{vmatrix}$$

$$= (b - a)(c - a)(c - b)$$

$$= (a - b)(b - c)(c - a)$$

Try to solve

7 Without expanding the determinant, prove that $\begin{vmatrix} a - b - c & 2b & 2c \\ 2a & b - c - a & 2c \\ 2a & 2b & c - a - b \end{vmatrix} = (a + b + c)^2$

Exercises 3 - 1

Choose thr correct answer from the given:

1. If $\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = 12$, then $\begin{vmatrix} z & x & y \\ f & d & e \\ c & a & b \end{vmatrix} =$

   a) -12 b) -6 c) 6 d) 12

2. If $\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = 15$, then $\begin{vmatrix} a & b & c \\ x & y & z \\ d & e & f \end{vmatrix} =$

   a) -30 b) -15 c) zero d) 15

3. $\begin{vmatrix} a & a & a \\ a & b & c \\ c & c & c \end{vmatrix} =$

   a) zero b) a c c) b c d) ab c

4. $\begin{vmatrix} ab & a & \frac{1}{c} \\ ac & c & \frac{1}{b} \\ b\,c & b & \frac{1}{a} \end{vmatrix} =$

   a) zero b) 1 c) 2 d) 5

5. $\begin{vmatrix} x - y & y - z & z - x \\ y - z & z - x & x - y \\ z - y & x - y & y - z \end{vmatrix} =$

   a) Zero b) x - y c) y-z d) z - x

6. If $i^2 = -1$, then $\begin{vmatrix} 1 & i & i + 1 \\ 0 & 1 & i - 1 \\ 0 & i & i \end{vmatrix} =$

   a) 2i - 1 b) 2i + 1 c) i d) 1

7) The solution set of the equation $\begin{vmatrix} x & 2x & 3x \\ 3x & 2x & x \\ x & -x & 0 \end{vmatrix} = 96$ in R is

a) 4 b) 3 c) 2 d) -2

8) In triangle ABC $\begin{vmatrix} a & b & c \\ 5 & 7 & 8 \\ \sin a & \sin b & \sin c \end{vmatrix} =$

a) 5a b) 7b c) 8c d) zero

9) If $n = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 3 & 5 \\ 1 & 4 & 2 \end{vmatrix}$, $m = \begin{vmatrix} 3 & 0 & 9 \\ 4 & 6 & 10 \\ 5 & 20 & 10 \end{vmatrix}$ then m =

a) n b) 10n c) 20n d) 30n

10) $\begin{vmatrix} 1 & -1 & 9 \\ 1 & 0 & -7 \\ -9 & 7 & 0 \end{vmatrix} =$

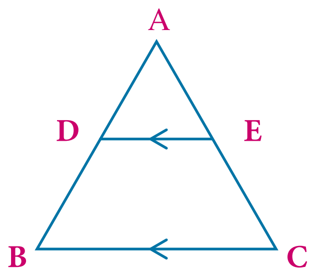
a) -9 b) -7 c) 49 d) zero

11) Prove that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)$

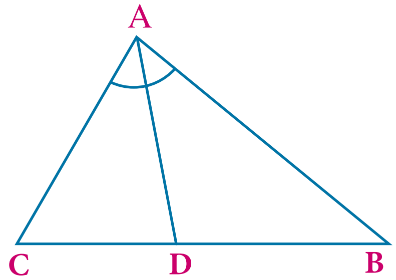
then find the numerical value of the determinant If $x - y = 5$, $y - z = 7$

12) **Geometry :** In the opposite figure $\overline{DE} \parallel \overline{BC}$

Prove that $\begin{vmatrix} 1 & 2 & 3 \\ DE & AD & AE \\ BC & AB & AC \end{vmatrix} = 0$



(13) In the opposite figure :



Prove that $\begin{vmatrix} 2 & 3 & 5 \\ AB & AC & AB+AC \\ BD & DC & BC \end{vmatrix} = 0$

(14) Prove that $\begin{vmatrix} \sin^2x & \cos^2x & 1 \\ \sin^2y & \cos^2y & 1 \\ \sin^2z & \cos^2z & 1 \end{vmatrix} = 0$

Use the properties of determinants to solve the following equations:

(15) $\begin{vmatrix} x & 1 & x \\ 2 & 3 & 4 \\ x & 5 & x \end{vmatrix} = 16$

(16) $\begin{vmatrix} 1 & -x & 0 \\ x & 1 & x \\ 1 & -1 & x+1 \end{vmatrix} = \begin{vmatrix} x^2 & 1 \\ -x & x \end{vmatrix}$

(17) $\begin{vmatrix} 2 & -x & 1 \\ 1 & -1 & 0 \\ x+1 & 1 & x \end{vmatrix} = 2x + 1$

(18) $\begin{vmatrix} x & 1 & 2 \\ 1 & x & 2 \\ 1 & 2 & x \end{vmatrix} = x - 1$

Use the properties of determinant to prove that:

(19) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-c)(c-b)$

(20) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(21) $\begin{vmatrix} bc & c^2 & b^2 \\ c^2 & ca & a^2 \\ b^2 & a^2 & ab \end{vmatrix} = a\,bc \begin{vmatrix} a & c & b \\ c & b & a \\ b & a & c \end{vmatrix}$

(22) $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 4xyz$

(23) **Without expanding the determinant, prove that** $\begin{vmatrix} 1 & 1 & 1 \\ x & y & x \\ x & x & -y \end{vmatrix} = x^2 - y^2$

Without expanding the determinant , find the value:

(24) $\begin{vmatrix} 5 & -1 & 10 \\ 4 & 2 & 8 \\ -5 & 2 & -10 \end{vmatrix}$

(25) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & -1 \\ -4 & 6 & 2 \end{vmatrix}$

Use the properties of determinants to prove that:

26) $\begin{vmatrix} (x+y)^2 & x^2+y^2 & 3x\,y \\ (z+L)^2 & z^2+L^2 & 3z\,L \\ (m+n)^2 & m^2+n^2 & 3m\,n \end{vmatrix} = 0$

27) $\begin{vmatrix} 0 & L & -m \\ -L & 0 & -n \\ m & n & 0 \end{vmatrix} = 0$

28) $\begin{vmatrix} 1 & m\,n & x\,y + z\,L \\ 1 & x\,y & m\,n + z\,L \\ 1 & z\,L & x\,y + m\,n \end{vmatrix} = 0$

Without expanding the detereminant , prove that:

29) $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x + 2a)(x - a)^2$

30) $\begin{vmatrix} 1 & x & y \\ x & 1+x^2 & x\,y \\ y & x\,y & 1+y^2 \end{vmatrix} = 1$

31) $\begin{vmatrix} x & -y & -z \\ y & 1 & 0 \\ z^2 & 0 & 1 \end{vmatrix} = x + y^2 + z^3$

32) Useg the properties of determinants ,

prove that $\begin{vmatrix} 2 & 4 & 3 \\ -5 & -1 & -7 \\ 1 & 5 & 2 \end{vmatrix} + \frac{1}{4} \begin{vmatrix} 4 & 6 & 2 \\ 8 & 6 & 10 \\ 6 & 9 & 4 \end{vmatrix} = 0$

33) Without expanding the determinant, prove that :

$$\begin{vmatrix} a & b & c \\ b & a & c \\ b & c & a \end{vmatrix} = (a - b)(a - c)(a + b + c)$$

34) Use the properties of determinants

prove that $\begin{vmatrix} b\,c & ab & 1 \\ ac & b^2 & b \\ b & c & a \end{vmatrix} = (a + b) \begin{vmatrix} 1 & b & 1 \\ a & b^2 & b \\ b & c^2 & c \end{vmatrix}$

35 Without expanding the determinant, prove that:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = a^2 + b^2 + c^2 + 1$$

36 Without expanding the determinant, prove that:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right)$$

37 If $\begin{vmatrix} a^3-1 & a^2 & a \\ b^3-1 & b^2 & b \\ c^3-1 & c^2 & c \end{vmatrix} = 0 \quad , a \neq b \neq c$

prove that abc=1

38 Without expanding the determinant , prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

39 Without expanding the determinant , prove that $\begin{vmatrix} bc & a^2 & a^2 \\ b^2 & ac & b^2 \\ c^2 & c^2 & ab \end{vmatrix} = \begin{vmatrix} bc & ab & ac \\ ab & ac & bc \\ ab & bc & ab \end{vmatrix}$

Matrices

Unit three

3 - 2

Preface

You have learned that the matrix is the set of elements arranged in m rows and n columns surrounded by the brackets in the form (   ) and symboled by one of the alphabet letters and the order of the matrix is written in the form $m \times n$

the matrix $A=\begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix}$ is a square matrix of the order 2×2

Also the matrix $B = \begin{pmatrix} 5 & 2 & 3 \\ 0 & 4 & -1 \\ 2 & 1 & 6 \end{pmatrix}$ is a square matrix of the order 3×3

while the matrix $C = \begin{pmatrix} 1 & -2 & 0 \\ 5 & 4 & 3 \end{pmatrix}$ is a matrix of the order 2×3

The matrices have applicable uses in linear transformation and in solving a system of linear equations. In addition, matrices are used in many fields of science such as mechanics, most of physics branches, graphs of computers, statistics and probability theorem.

Inveres of a matrix

You have studied how to find the multiplicative inverse (if exist) of a square matrix of the order 2×2 **and you knew that :**

If A, B are two square matrices of the order 2×2 **and if** $AB = BA = I$, **then** each of A and B are multiplicative inverse of the other .

We notice that some matrices have no multiplicative inverse.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the multiplicative inverse of A is defined when the determinant of $A = \triangle \neq 0$, then:

$$A^{-1} = \frac{1}{\triangle}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Remember

The identity matrix I is a square matrix all the elements of its principle diagonal are 1 and the other elements are zero

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

You will learn

- *The cofactor matrix*
- *The multiplicative inverse of a square matrix of the order 3×3.*
- *Solving linear equations using the multiplicative inverse of the matrix*
- *Homogeneous and non homogenous linear equations*
- *Rank of the matrix of coefficients*
- *Rank of augmented matrix*
- *The relation between the rank of the matrix of coefficients and the rank of the augmented matrix and possibilities of solutions*

Key terms

- *Inveres of a Matrix*
- *Adioient element*
- *Adjoint matrix*

Materials

- *Scientific calculator*

Example

1 Find the multiplicative inverse (if exist) for each of the following :

a $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$ b $B = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

Solution

a We find the determinant of the matrix $\triangle = \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 12 - 12 = 0$

$\therefore$ The matrix has no multiplicative inverse

b We find the determinant of the matrix $\triangle = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 6 - 4 = 2$

$\therefore$ The matrix B has multiplicative inverse denoted by B^{-1} where

$B^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$

Try to solve

1 Find the values of a which makes the matrix $\begin{pmatrix} a & 9 \\ 4 & a \end{pmatrix}$ have no multiplicative inverse.

> **Remember**
>
> **If $\triangle \neq 0$ then the matrix a has a multiplicative inverse determined as follows :**
> **a) interchange between the two positions of the two elements lying on the main diagonal of matrix a.**
> **b) Change both signs of the two elements lying on the other diagonal of matrix a**
> **c)Multiply the resulted matrix after doing (a), (b) by $\frac{1}{\triangle}$ to get a^{-1}**

The multiplicative inverse of a matrix of 3 × 3 system

If A is a matrix and $|A| \neq 0$, then A has a multiplicative inverse denoted by A^{-1} . A^{-1} a square matrix also $AA^{-1} = A^{-1}A = I$ where "I" is the identity matrix ,

> **Tip**
>
> **• The singular matrix (non-invertible) is the matrix which has no inverse $\triangle = 0$**
> **• The non singular matrix (invertible) is the magtrix which has inverse $\triangle \neq 0$**

Example

2 Determine whether the matrix A where $A = \begin{pmatrix} 3 & -1 & 2 \\ -1 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ has a multiplicative inverse or not. Explain.

Solution

We find the determinant of the square matrix as follows:

$$|A| = \begin{vmatrix} 3 & -1 & 2 \\ -1 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix}$$

$= 3(-2 - 0) + (1 - 1) + 2(0 - 2) = -6 - 1 - 4 = -10$

$\because$ $|A| \neq 0$, then A has a multiplicative inverse

Try to solve

2 Determine whether the matrix $B = \begin{pmatrix} 4 & -3 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has a multiplicative inverse or not.

The cofactors

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a matrix of the order 3 *3 and its determinant is $|A|$, then :

The cofactor of the element a_{ij} is the value of the minor determinant corresponding to the element a_{ij} which is formed by deleting the row and the column of a_{ij} and multiplying the result by $(-1)^{i+j}$ so the matrix of cofactors of the matrix A is given by :

$$C = \begin{pmatrix} (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

Example

3) Find the cofactors matrix of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}$

Solution

We find the cofactors of the matrix A as follows:

$C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2,\quad C_{21} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -2,\quad C_{31} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1$

$C_{12} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = 2,\quad C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1,\quad C_{32} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 0$

$C_{13} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5,\quad C_{23} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = 4,\quad C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$

We can summarize the sign rule which relate the minor determinants and the cofactors of any element of a square matrix as follows: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

So the cofactors matrix of A is : $C = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -1 & 0 \\ -5 & 4 & -1 \end{pmatrix}$

Try to solve

3) Find the cofactor matrix of the matrix A where $A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 4 & 5 \\ 3 & 6 & 7 \end{pmatrix}$

Adjoint matrix

The transpose of the cofactors matrix of the matrix A is called the adjoint matrix of the matrix A and denoted by Adj (A)

$$\text{Adj}(A) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}^t$$

In the previous example Adj(A) $= \begin{pmatrix} 2 & 2 & -5 \\ -2 & -1 & 4 \\ 1 & 0 & -1 \end{pmatrix}$

Example

4 Find the adjoint matrix of the matrix B where $B = \begin{pmatrix} 2 & 0 & 3 \\ -1 & 2 & -1 \\ 0 & 5 & 1 \end{pmatrix}$

Solution

We find the cofactors matrix of the matrix B

$$= \begin{pmatrix} \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 0 & 5 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 5 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 1 & -5 \\ 15 & 2 & -10 \\ -6 & -1 & 4 \end{pmatrix}$$

$$\therefore \ \text{Adj(B)} = \begin{pmatrix} 7 & 15 & -6 \\ 1 & 2 & -1 \\ -5 & -10 & 4 \end{pmatrix}$$

Critical thinking In the previous example, find each of : B × Adj (B) , Adj (B) × B. What do you notice?

Finding the inverse of 3 × 3 square matrix

To find the multiplicative inverse of the 3×3 matrix A using the cofactors matrix, we follow the next steps :

- Find the determinant of the matrix A , notice that $|A| \neq 0$
- Form the cofactors matrix for each element of the matrix A
- Find the adjoint matrix of A (the transpose of the cofactors matrix))
- Find the multiplicative inverse of A from the relation

$$A^{-1} = \frac{1}{|A|} \times \text{Adj (A)}$$

Example

5 Find the multiplicative inverse of the matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 3 & 5 \\ 1 & 4 & 0 \end{pmatrix}$

Solution

Find the determinant of A using the first row (say)

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 3 & 5 \\ 1 & 4 & 0 \end{vmatrix} = 2\,(0 - 20) + 1\,(0 - 5) + 3\,(16 - 3) = -40 - 5 + 39 = -6$$

Find the cofactors matrix $= \begin{pmatrix} \begin{vmatrix} 3 & 5 \\ 4 & 0 \end{vmatrix} & -\begin{vmatrix} 4 & 5 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 1 & 4 \end{vmatrix} \\ -\begin{vmatrix} -1 & 3 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} -1 & 3 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} \end{pmatrix}$

$= \begin{pmatrix} -20 & 5 & 13 \\ 12 & -3 & -9 \\ -14 & 2 & 10 \end{pmatrix}$

Find the adjoint matrix (the transpose of the cofactors matrix)

$Adj(A) = \begin{pmatrix} -20 & 12 & -14 \\ 5 & -3 & 2 \\ 13 & -9 & 10 \end{pmatrix}$ then $A^{-1} = \frac{1}{|a|} \times Adj(A) = \frac{1}{-6} \begin{pmatrix} -20 & 12 & -14 \\ 5 & -3 & 2 \\ 13 & -9 & 10 \end{pmatrix}$

Try to solve

4 Find the multiplicative inverse (if exist) for each of the following matrices:

$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 8 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 0 & 0 \\ 7 & 2 & -2 \\ 2 & -3 & 3 \end{pmatrix}$

Remember

The non-invertible matrix (signular) whose determinant = zero
the invertible matrix (nonsingular) whose determininat $\neq 0$

Some properties of inverse of a matrix

If A, B are two invertible (nonsingular) matrices, then:

1 $(A B)^{-1} = B^{-1} A^{-1}$

2 $(A^{-1})^{-1} = A$ **(the multiplicative inverse of the inverse of a matrix A = the matrix A))**

3 $(A^{-1})^{t} = (A^{t})^{-1}$ **(the transpose of the inverse = the inverse of the transpose))**

4 $(A^{-1})^{2} = (A^{2})^{-1}$ **(The square of the inverse = the inverse of the square)**

5 $(I)^{-1} = I$ **(The inverse of the identity matrix = the identity matrix)**

Example

6 If: $A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -1 & 5 \end{pmatrix}$, then verify each of the following properties :

First: $(A B)^{-1} = B^{-1} A^{-1}$ **Second:** $(A^{-1})^{-1} = A$

Solution

$|A| = 3 \times 2 - (-1) = 7$, $|B| = 2 \times 5 - 1(-1) = 11$

$A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$, $B^{-1} = \frac{1}{11}\begin{pmatrix} 5 & -1 \\ 1 & 2 \end{pmatrix}$

First:

$AB = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 0 & 11 \end{pmatrix}$

$\therefore |AB| = 77$

$\therefore (AB)^{-1} = \frac{1}{77} \begin{pmatrix} 11 & 2 \\ 0 & 7 \end{pmatrix}$ **(1)**

$B^{-1} A^{-1} = \frac{1}{11} \begin{pmatrix} 5 & -1 \\ 1 & 2 \end{pmatrix} \times \frac{1}{7} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$

$= \frac{1}{77} \begin{pmatrix} 11 & 2 \\ 0 & 7 \end{pmatrix}$ **(2)**

from (1) , (2) $\therefore (A B)^{-1} = B^{-1} A^{-1}$

Second:

$A^{-1} = \frac{1}{7} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-1}{2} & \frac{3}{7} \end{pmatrix}$

$\therefore |A^{-1}| = \frac{2}{7} \times \frac{3}{7} + \frac{1}{7} \times \frac{1}{7}$

$= \frac{6}{49} + \frac{1}{49} = \frac{1}{7}$

$\therefore (A^{-1})^{-1} = 7 \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = A$

$\therefore (A^{-1})^{-1} = A$

Try to solve

5. In the previous example, verify each of the following properties :

first: $(A^{-1})^t = (A^t)^{-1}$ **second:** $(A^{-1})^2 = (A^2)^{-1}$

Exercises (3 - 2)

First: Choose the correct answer from given answers:

1. The singular matrix from the following matrices is:

   a) $\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ b) $\begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ d) $\begin{pmatrix} 2 & 4 \\ -3 & 6 \end{pmatrix}$

2. The value of x which makes the matrix $\begin{pmatrix} x & 1 \\ -4 & 2 \end{pmatrix}$ is singular is:

   a) -2 b) $-\frac{1}{2}$ c) $\frac{1}{2}$ d) 2

3. All the following matrices have a multiplicative inverse except:

   a) $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 4 \\ 1 & -3 \end{pmatrix}$ c) $\begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

4. If A is nonsingular matrix, then $(AB)^{-1}$ equals:

   a) $- A B$ b) $A^{-1} B^{-1}$ c) $B^{-1} A^{-1}$ d) $(B A)^{-1}$

Second: Answer the following questions:

5. Find the value of x which makes each of the following matrices singular :

a $\begin{pmatrix} x-3 & 2 \\ 7 & x+2 \end{pmatrix}$ b $\begin{pmatrix} 3 & -1 & 3 \\ 2x & 3x & x+1 \\ 4 & 2 & 4 \end{pmatrix}$ c $\begin{pmatrix} x+4 & 2 & 1 \\ 3 & 4 & 5 \\ 7 & x-1 & 1 \end{pmatrix}$

6. Find the multiplicative inverse of each of the following matrices:

a $\begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$ b $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ c $\begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$ d $\begin{pmatrix} \sec\theta & \tan^2\theta \\ 1 & \sec\theta \end{pmatrix}$

e $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ f $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ g $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$ h $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

7. If $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$,verify that $(A\,B)^{-1} = B^{-1}A^{-1}$

8. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{pmatrix}$, verify that: $(A^t)^{-1} = (A^{-1})^t$

9. If $A = \begin{pmatrix} -3 & 4 & 0 \\ 2 & -1 & 5 \\ 1 & 0 & -2 \end{pmatrix}$, verify that: $(A^2)^{-1} = (A^{-1})^2$

10. **Creative thinking:** If $A = \begin{pmatrix} 3 & 2 \\ 10 & 4 \end{pmatrix}$, prove that $A^2 - 7A - 8\,I = \square$

then use this relation to find the multiplicative inverse of the matrix A

Unit three

3 - 3

Solving system of linear equations using matrix inverse

You will learn

- *Systems of linear equatrions*
- *Using the scientific calculator*
- *Converte the matrix equation into linear equations*
- *Find the matrix rank*
- *The possibilities of solving equations by the augmented matrix*

Key terms

- *Equation of a matrix*
- *Rank of a matrix*
- *Homogenuous equation*
- *Non Homgenuous equation*
- *Augmented Matrix*
- *Linear Equations*
- *Rank of a matrix*
- *Cofactor matrix*

Think and discuss

You have used algebraic methods to solve the following system of linear equations:

$$2x + 3y = 7 \quad , \quad 3x - y = 5$$

Can you represent the previous equations in the form of matrix equation and find the solution of these equation ?

The matrix equation represents a full system of equations

System of equations	matrix equation
$\begin{matrix} 2x + 3y = 7 \\ 3x - y = 5 \end{matrix}$	$\begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

- Compare the system of equation in the algebraic form and the system of the matrix equation then determine the coefficients matrix, the variables matrix and the constants matrix

Coefficients matrix (A)		Variables matrix (x)		Constants matrix (B)
$\begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}$	$\times$	$\begin{pmatrix} x \\ y \end{pmatrix}$	$=$	$\begin{pmatrix} 7 \\ 5 \end{pmatrix}$

The matrix equation can be written in the form: AX = B

- Can you find the product of the two matrices $\begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$?
- Compare the number of columns of the first matrix and the number of rows of the second matrix. What do you notice?
- The product you gained should be equal to $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$
- Can you use the inverse matrix in the previous lesson to solve the matrix equation?

Systems of linear Equations

We can solve «n» of linear equation in «n» variables which have unique solution using the matrices multiplications when n = 2 or n = 3
and considering the next system of equations:

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2 \qquad \text{we get } AX = B$$
$$a_3 x + b_3 y + c_3 z = d_3$$

where $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

$A^{-1}(A \times X) = A^{-1} \times B$ — **multiply both sides of the equation by A^{-1}**

$(A^{-1} \times A) \times X = A^{-1} \times B$ — **Associative property**

$I \times X = A^{-1} A$ — **because I is the identity element**

$X = A^{-1} B$

Note that: The solution of the matrix equation AX = B is the product of the inverse of the coefficients matrix and the constants matrix.

Example

1. Solve the following system of linear equation using the inverse matrix :
$4x + y = 0$, $x + 2z = 15$, $y - 7z = 0$

Solution

Write the matrix equation AX = B where

$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -7 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 15 \\ 0 \end{pmatrix}$

Find the inverse of the matrix A using the cofactors matrix

$|A| = \triangle = \begin{vmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -7 \end{vmatrix} = 4(-2) - 1(-7) = -1$

the cofactors matrix $= \begin{pmatrix} \begin{vmatrix} 0 & 2 \\ 1 & -7 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & -7 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 1 & -7 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ 0 & -7 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} \end{pmatrix}$

$$= \begin{pmatrix} -2 & 7 & 1 \\ 7 & -28 & -4 \\ 2 & -8 & -1 \end{pmatrix} \qquad Adj(A) = \begin{pmatrix} -2 & 7 & 2 \\ 7 & -28 & -8 \\ 1 & -4 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|a|} \times Adj\,(A) = \frac{1}{-1}\begin{pmatrix} -2 & 7 & 2 \\ 7 & -28 & -8 \\ 1 & -4 & -1 \end{pmatrix} \qquad \therefore X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} -2 & 7 & 2 \\ 7 & -28 & -8 \\ 1 & -4 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 15 \\ 0 \end{pmatrix} = \begin{pmatrix} 105 \\ 480 \\ 60 \end{pmatrix}$$

$\therefore$ x = -105 , y = 480 , z = 60 The solution set = {(-105 , 480 , 60)}

Try to solve

1. Solve the following system of equations:
   2x - 3y - z =9 , x + 2y + 3z = 15 , x-2z =12 using the inverse matrix

Using the scientific calculator

We can use the scientific calculator to solve system of linear equations in three variables as follow:

Press the botton MODE and the next tool list will be appeared:

MODE 1 (cOMP)	General calculations
MODE 2 (cMPLX)	complex number calculations
MODE 3 (STAT)	Statistical and regression calculations
MODE 4 (BASE-N)	calculations involving specific number systems (binary, octal, decimal, hexadeiaml)
MODE 5 (EQN)	Equation solution
MODE 6 (MATRIX)	Matrix calculations
MODE 7 (TABLE)	Generate a number table based on one or two functions.
MODE 8 (VEcTOR)	Vector calculations
MODE ▼ 1 (INEQ)	Inequality solution
MODE ▼ 2 (VERIF)	Verify a calculation
MODE ▼ 3 (DIST)	Distribution calculations

- Select [5] from (EQN) and the next tool list will be appeared:

Press this key:	*To select this calculation Type:*
[1] $(a_n x + b_n y = c_n)$	Simultaneous linear equations with two unknowms
[2] $(a_n x + b_n y + c_n z = d_n)$	Simultaneous linear equations with three unknowms
[3] $(ax^2 + bx = 0)$	Quadratic equation
[4] $(ax^3 + bx^2 + cx + d = 0)$	cubic equation

- Select [2] and press on it and let the equations be :

$x - y + z = 2$, $x + y - z = 0$, $-x + y + z = 4$

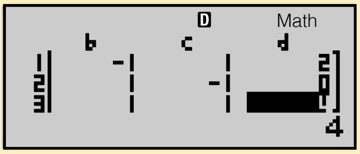
$x - y + z = 2,\ y - z = 0,\ -x + y + z = 4$

[MODE] [5] (EQN) [2] $(a_n X + b_n Y + c_n Z = d_n)$

1 [=] [(-)] 1 [=] 1 [=] 2 [=]

1 [=] 1 [=] [(-)] 1 [=] 0 [=]

[(-)] 1 [=] 1 [=] 1 [=] 4 [=]



[=] (X=) = 1

[▼] (Y=) = 2

[▼] (Z=) = 3

System of homogeneuous and non - homogenous linear equations

We say that the system of linear equations is homogeneous if all the elements of the constants matrix equals zero

i.e. $B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, but if one of the elements of the constants matrix is not equal to zero, then the system of linear equation is non- homogeneuous

Verbal expression: Show which of the following systems represents a homogeneuous system of linear equations and which represents a non-homogeneuous system of linear equations.

1 $3x + 2y - 5z = 0$, $5x - 3y + 2z = 4$, $x - 2z = 0$

2 $2x + 3y = 5z$, $3x + z = 4y$, $x + z = 0$

Rank of a matrix

Definition

The rank of the non-zero matrix is the greatest order of determinant or minor determinant of the matrix whose value does not vanish so if A is a non-zero matrix of the order $m \times n$ where $m \geqslant n$,then the rank of the matrix (A) is denoted by RK (A) where $1 \leqslant RK(A) \leqslant n$.

Example

2 Find the rank of each of the following matrices $A = \begin{pmatrix} 3 & -2 & 4 \\ 2 & -3 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 & 5 \\ 3 & 12 & 15 \end{pmatrix}$

Solution

First: The matrix $A = \begin{pmatrix} 3 & -1 & 4 \\ 2 & 5 & 1 \end{pmatrix}$ of the order 2×3, then the greatest order of a minor determinant can be formed from A is 2

$\therefore \begin{vmatrix} 3 & -2 \\ 2 & -3 \end{vmatrix} = -9 + 4 = -5 \neq 0$ $\qquad \therefore RK(A) = 2$

Second: the matrix $B = \begin{pmatrix} 1 & 4 & 5 \\ 3 & 12 & 15 \end{pmatrix}$ of the order 2×3, then the greatest order of a minor determinant can be formed from B is 2

$\begin{vmatrix} 1 & 4 \\ 3 & 12 \end{vmatrix} = 0$, $\begin{vmatrix} 4 & 5 \\ 12 & 15 \end{vmatrix} = 0$, $\begin{vmatrix} 1 & 5 \\ 3 & 15 \end{vmatrix} = 0$

$\because$ the value of each minor determinant $= 0$

$\therefore RK(A) < 2$ $\qquad \therefore RK(A) = 1$

Try to solve

2 Find the rank for each of the following matrices:

$A = \begin{pmatrix} 2 & 7 & -3 \\ 3 & 5 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1,5 & 3 & 9 \\ 1 & 2 & 6 \end{pmatrix}$

Example

3 Find the rank for each of the following matrices: $A = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & -1 \\ 2 & 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 3 & 5 \\ -1 & 2 & 1 \end{pmatrix}$

Solution

$|A| = \begin{vmatrix} 2 & 0 & 3 \\ 1 & 5 & -1 \\ 2 & 3 & 4 \end{vmatrix} = 25 \neq 0 \quad \therefore RK(A) = 3$

$|B| = \begin{vmatrix} 3 & 1 & 4 \\ 2 & 3 & 5 \\ -1 & 2 & 1 \end{vmatrix} = 0 \quad (R_3 = R_1 + R_2)$ $\qquad \therefore RK(B) < 3$

We find the minor determinant of order 2

$\begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = 3 - 10 = -7 \neq 0$ $\qquad \therefore RK(B) = 2$

Try to solve

3 Find the rank of the following matrices $A = \begin{pmatrix} 5 & 2 & 0 \\ 3 & 1 & 0 \\ 1 & -1 & 3 \end{pmatrix}$, and $B = \begin{pmatrix} 2 & 3 & 5 \\ 7 & 4 & -2 \\ 6 & 9 & 15 \end{pmatrix}$

Note that

1- If I is the identity matrix of the order $m \times m$, then the rank of I equals m because $|I| = 1 \neq 0$

2- The rank of the zero matrix = zero

3- The rank of the matrix A = the rank of the matrix A^t

4- If a zero row (column) is added (deleted) to the matrix A , then the rank of A doesn't change .

5- if the sum of rows (columns) is added (deleted) to the matrix A , then the rank of A doesn't change .

Critical thinking

1- If the matrix A $= \begin{pmatrix} 2 & 1 & 3 \\ k & 0 & 1 \\ 2 & 4 & -1 \end{pmatrix}$ and RK (A) = 2, find the value of K

2- If the matrix A $= \begin{pmatrix} -1 & 3 & 1 \\ 0 & k & 2 \\ 3 & 1 & 4 \end{pmatrix}$ and RK (A) = 3, find the real value of K

The augmented matrix

If we have (m) linear equations in (n) variables, then we write AX = B and we define the augmented matrix A as $A^* = (A|B)$ of the order $m \times (n+1)$

Example

4 Find the augmented matrix for each of the following systems:

a $3x - 5y = 2$, $2x + 7y = 9$, $4x - y = 3$

b $x + y + z = 9$, $2x - 3y + 2z = 3$, $3x + 2y - 3z = -3$

Solution

a $A^* = \left(\begin{array}{cc:c} 3 & -5 & 2 \\ 2 & 7 & 9 \\ 4 & -1 & 3 \end{array}\right)$

b $B^* = \left(\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 2 & -3 & 2 & 3 \\ 3 & 2 & -3 & -3 \end{array}\right)$

Try to solve

4 Find the augmented matrix for each of the following systems:

a $2x + 3y = 7$, $3x - y = 5$, $x - y = 1$

b $3x + 2y - z = 4$, $x + y + z = 3$, $x - z = 0$

Example

5 Find the rank of the augmented matrix for the system 2x -y = 3 , 6 x - 3y = 9

Solution

$A^* = \left(\begin{array}{cc:c} 2 & -1 & 3 \\ 6 & -3 & 9 \end{array}\right)$ is a matrix of the order 2×3

$\therefore$ The greatest order of a determinant can be formed from A* is 2

$\begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} = -6 + 6 = 0$, $\begin{vmatrix} -1 & 3 \\ -3 & 9 \end{vmatrix} = -9 + 9 = 0$, $\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 18 - 18 = 0$

$\because$ the value of all of the minor determinants = 0

$\therefore$ RK (A*) < 2 RK (A^*) = 1

Try to solve

5 Find the rank of the augmented matrix for each of the following systems

a 3x + 2y= 4 , 2x + 3y = -6

b 3x - 5y= 2 , 9x + 15y = 10

Possibility of solving a system of linear equatins

First: The non-homogeneous equations

The equation : $a_1x_1 + a_2x_2 + \dots\dots + a_n x_n = c$ is called a non-homogeneous equation where $c \neq 0$

and the system AX = C is called a non-homogeneous system where $C \neq \square$

1- The system of (n) equations in (n) variables has a unique solution if
RK (A) = RK(A^*) = n

2- The system has infinite number of solutions if
RK (A) = RK(A^*) = k where k < n

3- the system has no solution if RK(A) $\neq$ RK(A^*).

Linear equation in three variables

let the system of the equations be

$a_1 x + b_1 y + c_1 z = k_1$ $a_2 x + b_2 y + c_2 z = k_2$

$a_3 x + b_3 y + c_3 z = k_3$ **i.e.** AX = C

where

$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $C = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$ $A^* = \begin{pmatrix} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \end{pmatrix}$

The following table summarizes the possibility of solving the previous system (if $k_1 = k_2 = k_3 = 0$ (homogeneous equation - the rank of the augmented matrix doesn't effect)

RK(A*)	RK(A)	Possibilities of solution
3	3	unique solution
3	2	no solution
3	1	no solution
2	2	infinite number of solutions
2	1	no solution
1	1	infinite number of solutions

Second: the Homogeneous equations

The equation: $a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$ is called a linear homogeneous equation. The system of homogeneous linear equation is written in the form of AX = ☐ and the homogeneous equations are different from the non-homogeneous equations by the rank of the coefficients matrix A which is the same rank of the augmented matrix A^*:

1- If RK (A) = RK(A^*)= n (number of variables), then the system has a unique solution $x_1 = x_2 = x_3 = \dots = x_n = 0$
this solutions is called zero solution (Trivial solutions)

2- If the rank of the coefficients matrix is less than the number of variables n (RK(A) < n , $|A| = 0$) then the system has infinite number of solutions other than the zero solution

Example

6) Show that the system $3x - 2y + 3z = 0$, $x + y = 0$, $y - z = 0$ has a zero solution only .

Solution

$$A = \begin{pmatrix} 3 & -2 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$ is a square matrix of the order 3×3

$$\therefore |A| = \begin{vmatrix} 3 & -2 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$ using the elements of the first column

$\therefore |A| = 3(-1 - 0) - 1(2 - 3) + 0(0 - 5)$

$|A| = -3 + 1 = -2 \neq 0$ $\therefore$ RK (A) = 3 = number of variables

$\therefore$ the system has a zero solution as a unique solution
$x = 0, y = 0, z = 0$ the solution set = {(0,0 , 0)}

Try to solve

6) Show that the system $2x + y - z = 0$, $x - z = 0$, $2y + z = 0$ has a zero solution only.

Example

7 Show that the system $3x + y + 4z = 0$, $2x + 3y + 5z = 0$, $-x + 2y + z=0$ has infinite number of solutions and find the general form of the solution.

Solution

$$A = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 3 & 5 \\ -1 & 2 & 1 \end{pmatrix} \text{ is a square matrix of the order } 3 \times 3$$

$$\therefore |A| = \begin{vmatrix} 3 & 1 & 4 \\ 2 & 3 & 5 \\ -1 & 2 & 1 \end{vmatrix} = 0, \text{ where } R_3 = R_1 + R_2 \qquad \therefore RK(A) = 2$$

$\because$ RK (A) =2 (less than 3) number of variables

$\therefore$ the system has infinite number of solutions in the form (t, t , -t).

Ask your teacher about how to find the previous form of the general solution

Try to solve

7 Show that the system of equations $2x+ 3y+ 5z = 0$, $7x + 4y -2z=0$, $6x + 9y + 15z = 0$ has an infinite number of solutions and find the general form of the solution.

Exercises (3 - 3)

First: Choose the correct answer from the given:

1. Of the following linear systems , the homogeneous equations are:
   a) $2x + y = 3$, $x + 2y = 2$
   b) $x - y = 0$, $x + 2y = 12$
   c) $3x + y = 1$, $2x + y = 0$
   d) $x - 2y = 0$, $3x + y = 0$

2. If $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, then $\begin{pmatrix} x \\ y \end{pmatrix} =$
   a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
   b) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
   c) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
   d) $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

3. If $A = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 8 & 16 \end{pmatrix}$, then RK(A) =
   a) zero
   b) 1
   c) 2
   d) 3

4. The rank of the identity matrix I_3 is
   a) 3
   b) 2
   c) 1
   d) zero

5. the rank of the zero matrix of the order 3×3 is
   a) zero
   b) 1
   c) 2
   d) 3

6. If $A = \begin{pmatrix} 1 & -2 & 3 \\ m & 0 & 1 \\ 3 & 2 & -1 \end{pmatrix}$ and if RK (A) = 2, then m =
   a) -2
   b) zero
   c) 2
   d) 6

7. If m is the number of linear equations and n is the number of variables, then the augmented matrix is of the order
   a) $m \times n$
   b) $m \times (n + 1)$
   c) $(m + 1) \times n$
   d) $(m + 1) (n + 1)$

8. The rank of the augmented matrix of the system $-3y = 5$, $6x - 9y = 15$ is
   a) zero
   b) 1
   c) 2
   d) 3

9. The number of solutions of the system: $2x + 5y = 0$, $3x - z = 0$, $2y - 3z = 0$ is
   a) the zero solution only.
   b) zero.
   c) finite number of solutions except the zero solution.
   d) infinite number of solutions other than the zero solution.

10 the system $\begin{pmatrix} 2 & 2 & 3 \\ 1 & -2 & -3 \\ 4 & -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \square$ has

- **a** The trivial solution only
- **b** Infinite number of solutions other than the zero solution .
- **c** Finite number of solutions except the zero solution
- **d** No solution at all.

Answer the following questions:

11 Solve each of the following matrix equations:

a $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

d $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix}$ **e** $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ **f** $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}$

12 Write the augmented matrix of each of the following systems , then use the inverse matrix (if possible) to solve each system :

- **a** $2x + y + z = 0$, $3x + 2y + 3z = 12$, $4x + y + 2z = -1$
- **b** $x + 3y + 2z = 0$, $x + z = -1$, $x + 2y = 3$
- **c** $2x + y + 3z = 0$, $x + y = -1$, $y + 2z = 3$
- **d** $4x + 3y - 5z = 6$, $3x + 2y + 4z = 12$, $x - y + z = 2$

13 Show that each of the following systems has the zero solution only:

- **a** $2x + 7y + 3z = 0$, $3x + y - 2z = 0$, $4x - 3y - z = 0$
- **b** $x - 2y + 2z = 0$, $3x + 4z = 0$, $6z - y = 0$
- **c** $x + 2y - z = 0$, $2x + 3y - z = 0$, $3x - 4y + 2z = 0$

14 Show that each of the following systems has infinite number of solutions and write the general form of the solution .

- **a** $x + 2y + 3z = 0$, $2x + 3y + 5z = 0$, $3x - y + 2z = 0$
- **b** $2x - y + 3z = 0$, $4x - 2y + 6z = 0$, $x + 2z = 0$
- **c** $3x - 5y - 2z = 0$, $2x + y + 3z = 0$, $-x + 3y + 2z = 0$

Unit Summary

The determiniant: the determinant of the order n consists of n rows and n columns and it is formed by eliminating (n-1) variable from n of linear equations

Properties of determinants:

- In any determinant , if the rows are replaced by the columns and the columns are replaced by the rows in the same order, then the value of the determinant is unchanged.
- The value of a determinant does not change by evaluating it in terms of the elements of any of its rows (columns).
- If there is a common factor in all the elements of any row (column) in a determinant, then this factor can be taken outside the determinant.
- The value of the determinant is equal to zero in each of the following cases:

✔ If all the elements of any row (column) in a determinant are zeros , then the value of the determinant is zero.

✔ If the corresponding elements in two rows (columns) of any determinant are equal , then the value of the determinant is zero.

- If the positions of two rows (columns) are interchanged , then the value of the resulted determinant is equal to the value of the original determinant multiplies by (-1) .
- if all the elements of any row (column) are written as the sum of two elements, then the value of the determinant can be written as the sum of two determinants .
- If we add to all the elements of any row (column) a multiple of the elements of another row (column), the value of the determinant is unchanged.
- The value of the determinant in the triangular form is equal to the product of the elements of its main diagonal.

To find the inverse of a 3×3 square matrix , we follow the next steps:

- Find the determinant of the matrix A where $|A| \neq 0$
- Form the cofactors matrix (C) of elements of the matrix A .
- Find the adjoint matrix of A (the transpose of the cofactors matrix) .
- Find the multiplicative inverse of the matrix A using the relation : $A^{-1} = \frac{1}{|A|} \times \text{Adj}(A)$

Unit summary

Solving systems of linear equations

Considering

A is the coefficients matrix, X is the variables matrix ,

B is the constants matrix, then

- The matrix equation is written in the form AX: B
- The solution os this equation is : $X = A^{-1} \times B$

The rank of the matix:

the rank of the non-zero matrix is the greatest order of determinant or minor determinant of the matrix whose value does not vanish, so if A is a non-zero matrix of the order $m \times n$ where $m \geqslant n$, then the rank of the matrix A is denoted by RK (A) where $1 \leqslant$ RK (A) $\leqslant n$

The augmented matrix:

It is an extended matrix for a linear system and denoted by A^* where:

$A^* = (A \mid B)$ is of the order $m \times (n + 1)$.

Non–homogeneous equations

The system of equations in the form of matrix equation : A X = B is said to be non-homogeneous where $B \neq \square$

the system of (n) equations in (n) variables has a unique solution if RK(A) = RK(A*) = n , $|A| \neq 0$

- the system has infinite number of solutions if RK(A) = RK $(A^{\times})$ = k Where $K < n$
- The system has no solution if RK(A) $\neq$RK$(A^{\times})$

Homogeneous equations:

The system of equations in the form : $A X = \square$ are called homogeneous equations and if:

RK(A) = RK(A^*) = n (number of variables), then the system has a unique solution which is the zero solution (trivial solution)

RK(A) $<$ n (number of variables) , $|A| = 0$, then the system has solutions other than the zero solution .

General exercises

First L Choose the correct answer from given:

1. $\begin{vmatrix} 24 & 25 & 26 \\ 27 & 28 & 29 \\ 30 & 31 & 32 \end{vmatrix} =$ ....

   a) zero b) 12 c) 24 d) 56

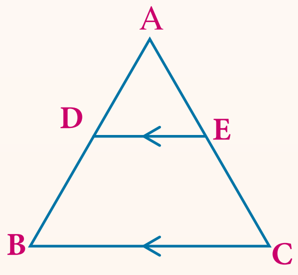
2. The solution set of the equation $\begin{vmatrix} 2 & 5 & 0 \\ 4 & x & 0 \\ x & 7 & 5 \end{vmatrix} = 0$ is ....

   a) {2} b) {5} c) {7} d) {10}

3. $\begin{vmatrix} a+b & c+b & a+c \\ c & a & b \\ 1 & 1 & 1 \end{vmatrix} =$ ....

   a) -1 b) zero c) ∀ + b + c d) ∀b c

4. in the opposite figure $\overline{DE} \parallel \overline{BC}$



   $\begin{vmatrix} 5 & 6 & 7 \\ DE & AD & AE \\ BC & AB & AC \end{vmatrix} =$ ....

   a) 7 b) 6 c) 5 d) zero

5. The value of x which makes the matrix $\begin{pmatrix} x-1 & 2 \\ 4 & x+1 \end{pmatrix}$ singular is:

   a) - 3 b) 3 c) ± 3 d) 9

6. each of the following matrices has an inverse except :

   a) $\begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ c) $\begin{pmatrix} 4 & -8 \\ 2 & -4 \end{pmatrix}$ d) $\begin{pmatrix} 4 & -2 \\ 10 & 5 \end{pmatrix}$

7. if the matrix A $= \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 6 \\ 3 & -6 & -9 \end{pmatrix}$ then RK(A) =

a) zero b) 1 c) 2 d) 3

Second : Answer the following questions :

Without expanding the determinant , prove that :

8. $\begin{vmatrix} 1 & b & c \\ b & 1+b^2 & bc \\ c & bc & 1+c^2 \end{vmatrix} = 1$

9. $\begin{vmatrix} 1 & bc & ca+ab \\ 1 & ca & ab+bc \\ 1 & ab & bc+ca \end{vmatrix} = \text{zero}$

10. $\begin{vmatrix} x+y+2 & x & y \\ 1 & 2x+y+1 & y \\ 1 & x & x+2y+1 \end{vmatrix} = 2(x+y+1)^3$

11. $\begin{vmatrix} 1 & 1 & x \\ -2x+2x^2 & 1+2x^2 & 2x^3 \\ -3+2x^2 & 3x^2 & 1+3x^3 \end{vmatrix} = 1+5x+x^3$

12. $\begin{vmatrix} x+c & x+c & c^2 \\ c & x & c \\ 0 & c-x & x-c \end{vmatrix} = (x+2c)(x-c)^2$

13. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

14. If (x – 1) is a factor of the determinant $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & 1 & x+1 \\ -1 & 1 & x+k \end{vmatrix}$, find the value of x

15. If x is a factor of the determinant $\begin{vmatrix} x+c & 3 & 1 \\ k & 2 & 3 \\ x & 2 & 1 \end{vmatrix}$, find the value of k

Third : Discuss the possibility of solving of each of the following linear equations and find the solutions if exist:

16 $x + 2y + z = 3$, $4x - y - z = 6$, $x + y + 3z = 10$

17 $x + y + z = 1$, $2x - y + 3z = -5$, $3x + 2y + 3z = 0$

18 $x + 2y - 3z = 0$, $z - x - 2y = 0$, $2x + 4y - 6z = 0$

19 $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

20 $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -3 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$

21 $\begin{pmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ -3 & 6 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

For more activities and exercise, visit www.sec3mathematics.com.eg

Accumulative test

Choose the correct answer from the given:

1. If $\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} = 2$, then $\begin{vmatrix} 5a & b & c \\ 5d & e & f \\ 35x & 7y & 7z \end{vmatrix} =$

   a) 700 b) 10 c) 35 d) 70

2. If $A = \begin{pmatrix} 2 & 5 \\ 5 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -4 & 3 \\ 1 & 1 \end{pmatrix}$ and if $A \times C = B$, then $C =$

   a) $\begin{pmatrix} -8 & 1 \\ 3 & 2 \end{pmatrix}$ b) $\begin{pmatrix} -11 & 4 \\ 3 & -1 \end{pmatrix}$ c) $\begin{pmatrix} -17 & 4 \\ 6 & -1 \end{pmatrix}$ d) $\begin{pmatrix} -9 & 6 \\ 3 & -1 \end{pmatrix}$

3. the matrix $\begin{pmatrix} 2 & a \\ a & 8 \end{pmatrix}$ is singular if a =

   a) -4 b) 4 c) ±4 d) 16

4. If $A^{-1} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and if $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then y =

   a) 5 b) 6 c) 7 d) 8

5. $\begin{vmatrix} c+b & 5 & c \\ b+c & 5 & c \\ c+c & 5 & b \end{vmatrix} = ...$

   a) 5 b) 4 c) 3 d) zero

6. If the equations $x + 2y + 3z = 5, 2x - 3y + kz = 13$ $3x + ky + 2z = 3$ have a unique solution, then $k \in$

   a) R b) R - {-1} c) R - {13} d) R - {-1, 13}

7. If $A = \begin{pmatrix} 2 & 3 & -1 \\ -2 & -3 & 1 \\ 6 & 9 & -3 \end{pmatrix}$, then RK(A) =

   a) zero b) 1 c) 2 d) 3

8. If $A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 0 & 2 \\ 2 & -2 & 1 \end{pmatrix}$, then RK(A^t) =

   a) zero b) 1 c) 2 d) 3

Second : Answer the following questions:

9 Without expanding the determinant, prove that: $\begin{vmatrix} 3x & 3x & 3x \\ 1 & b & a \\ a+b & a+1 & b+1 \end{vmatrix} = 0$

10 Find the rank of the matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 4 & -5 \\ 4 & 5 & -1 \end{pmatrix}$

11 Without expanding the determinant , prove that $\begin{vmatrix} 1 & 1 & 1 \\ 1+y & 1 & 1 \\ 1 & 1+y & 1 \end{vmatrix} = y^2$

12 Use the inverse of the matrix to solve the following system of equations:

$2x + 2y - z = 3,\ 3x + y = 5,\ x + y + 2z = 9$

13 Use properties of determinant to prove that:

$\begin{vmatrix} a+x & x & x \\ x & b+x & x \\ x & x & c+x \end{vmatrix} = a\,bc + x\,(ab + bc + a\,c)$

14 Discuss the possibility of solving the following system of equations:

$x - y + z = 2\ ,\ 2x + 3y - z = 5,\ 3x - 5y + 2z = -1$

15 If $\begin{vmatrix} x & y & z+2 \\ x & y+2 & z \\ x+2 & y & z \end{vmatrix} = -4$ find the value of $x + y + z$

16 Show which of the following matrices is singular and which is nonsingular:

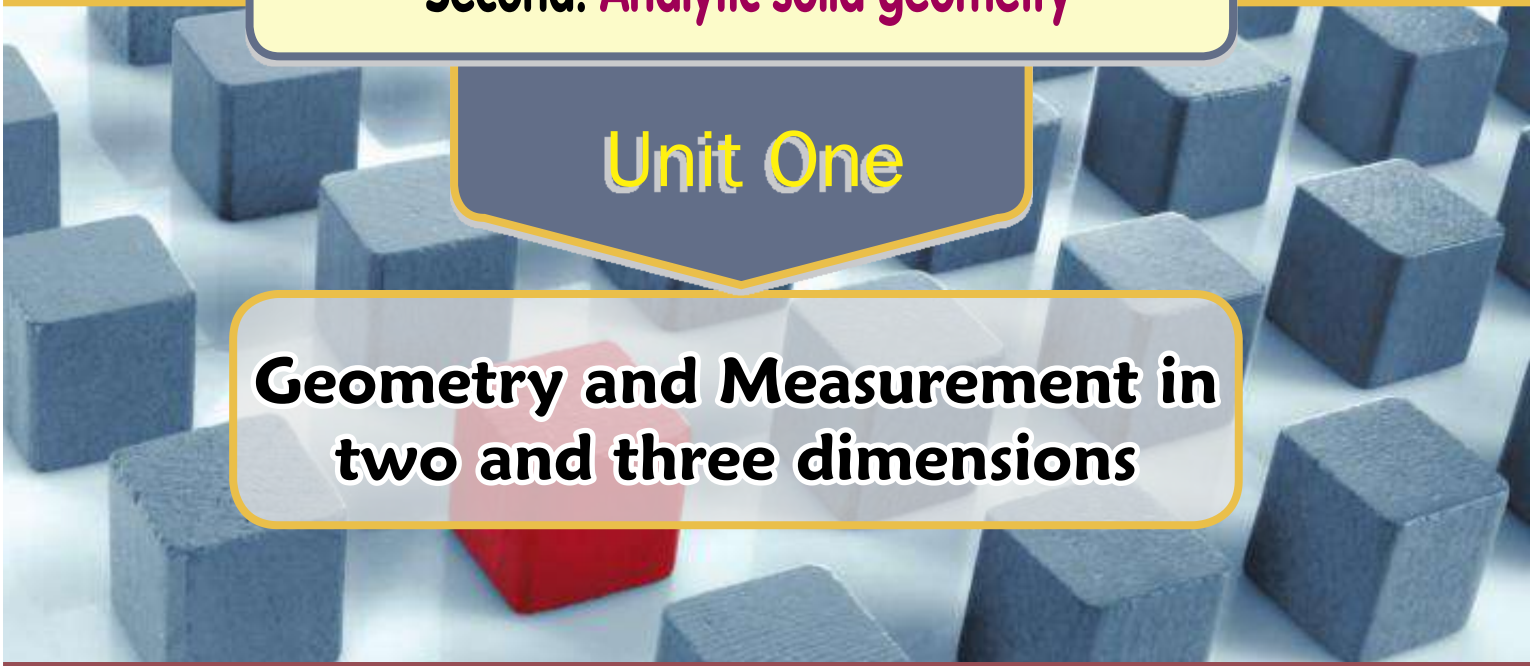
a $A = \begin{pmatrix} 3 & 1 \\ -2 & 5 \end{pmatrix}$

b $B = \begin{pmatrix} 8 & 16 \\ 2 & 4 \end{pmatrix}$

c $C = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 2 & 0 & -4 \end{pmatrix}$

d $D = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{pmatrix}$

Second: Analytic solid geometry



Unit One

Geometry and Measurement in two and three dimensions

Unit introduction

Geometry is the scince of studying the various types of shapes and describing them. It studies the relation of the shapes, angles and distances with each other. Besides, it is divided into two parts: The plane geometry which is specialized in studying the geometric shapes of 2-D only and the solid geometry which is specialized in studying the solids of 3-D (length - width- height). It deals with spaces such as cuboids, cylindric solids, spherical and conical bodies.

Greek had been the first to use geometry. Thales had discovered proofs for some theorems, then Euclid had assembled all the geometric corollaries and organized them in a book called "the principles". Later , this type of geometry have been developed into the analytic geometery, geometry of triangles, Minkowski geometry (4-D) and the non-Euclidean geometry and so on. In this unit, you learn the vectors to study the straight lines, planes and the relation among them in 3-D

Unit objectives

By the end of this unit and doing all activities included, the student should be able to:

- Recognize the coordinate system in three dimensions and resolve the vector in space.
- Find the distance between two points in space and the coordinate of the midpoint of a line segment in space .
- Find the cartesian equation of the sphere in terms of the coordinates of the centre and a point on the sphere
- Recognize the vectors in space through:
  - Representing the vector by triple ordered.
  - The fundamental unit vectors in space $\hat{j} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$.
  - Expressing any vector interims of $\hat{i}$, $\hat{j}$, $\hat{k}$
  - Expressing directing line segment in terms of its ends in space.
- Recognize the scalar product and vector product of two vectors in plane and space.
- Recognize the properties of the scalar product and vector product in plane and space.
- Recognize the angle between two vectors in space.
- Recognize the perpendicularity of two vectors in space
- Identify the direction angles and direction cosines of any vector in space.
- Use the scalar product to find the algebraic and vector companents of avector in direction of another vector
- Use the scalar product to find the work done.
- Recognize the geometrical meaning of the norm of the vector product.
- Recognize the scalar triple product and its geometrical meaning.

Key terms

- space
- 3D
- projection
- right hand Rule
- 3D−vector
- plane
- scalar product
- vector product
- component
- work
- scalar triple product
- position vector
- unit vector
- the norm of vector

Materials

- Scientific calculator

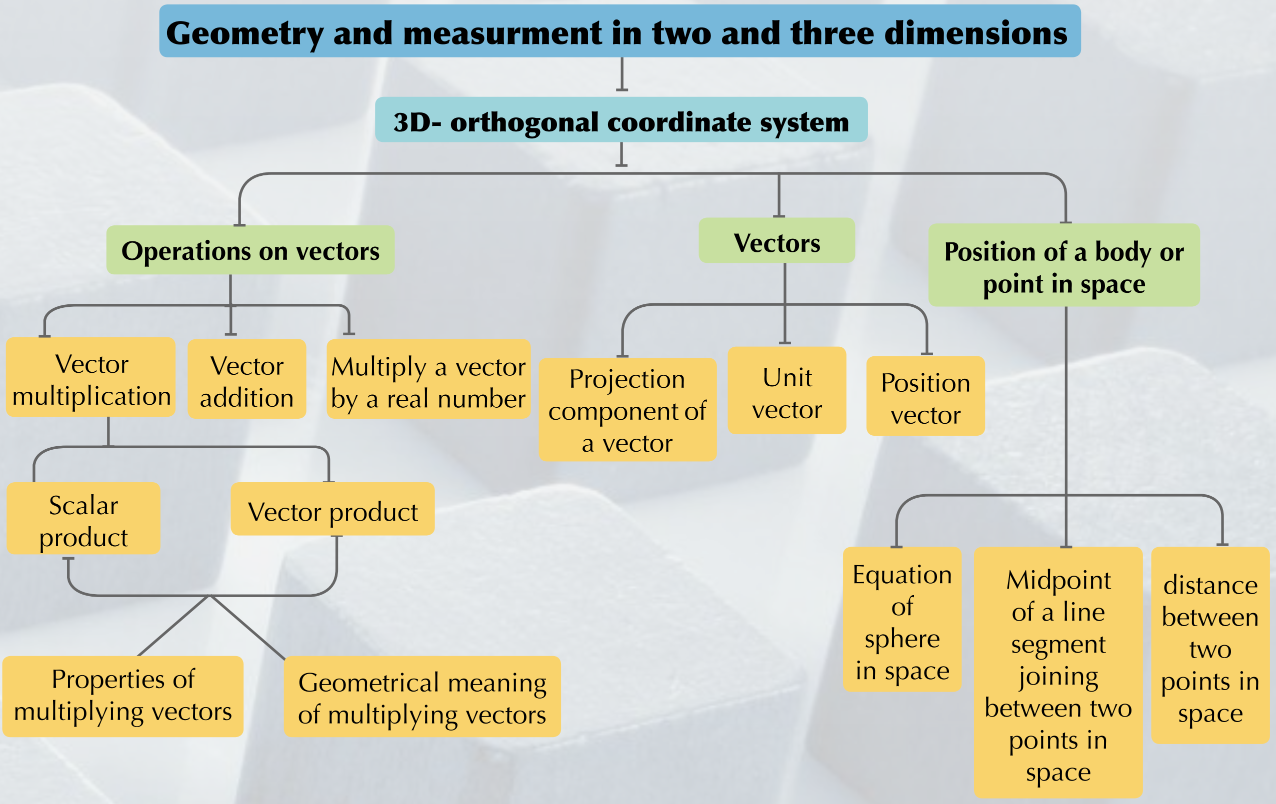
Unit Lessons

Lesson (1 - 1): The 3D orthogonal coordinate system.

Lesson (1 - 2): vectors in the space.

Lesson (1 - 3): Vector multiplication.

Unit chart



Unit One

1 - 1

The three- dimensional orthogonal coordinate system

You will learn

- Determine the position of a point in 3D-coordinate system
- Identify the coordinates of a midpoint of a line segment joining between 2 points in space
- Find the distance between 2 points in the space
- Equation of the sphere in space

Key terms

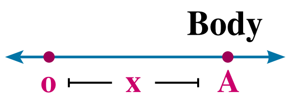
- Space
- 3D
- Projection
- Right hand rule
- Plane

Materials

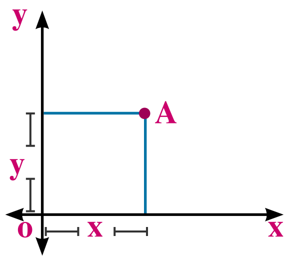
- Scientific calculator

Think and discuss

- To determine the position of a body on a straight line, we should know the distance between the body and a fixed (arbitrary) point on it called the origin (O)
  OA = x ∈ R



- To determine the position of a body in plane, you should know its projection on two dimensional orthognal axis.
  A = (x, y) ∈ R^2

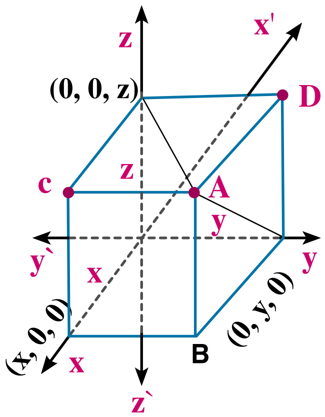


- How can you determine the position of a body in space?

Learn

The three - dimensional orthogonal coordinate system R^3

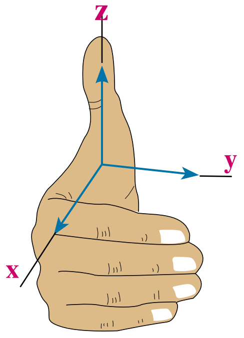
The coordinates of a point A in space are identified with respect to 3 mutually orthogonal axes intersect at a point by finding the projection of this point on each axis .



Think: In the previous 3 D coordinate system, find the coordinates of the points B, C, D,

basic definitions:

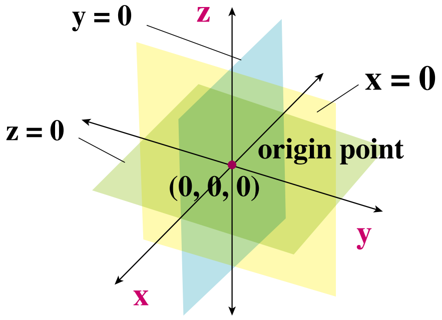
1- The right hand rule



when the 3-dimensional orthogonal coordinate system is formed, we should follow the right hand rule.
where the curved fingers refer from the +ve direction of x-axis towards the positive direction of y-axis, your thumb points at the direction of positive z-axis.

2- Cartisian planes

- All points in space with coordinates (x, y, 0) are located in x y plane whose equation is $Z = 0$
- All points in space with coordinates (x, 0, z) are located in x z plane whose equation is $y = 0$
- All points in space with coordinates (0, y, z) are located in y z plane whose equation is $x = 0$

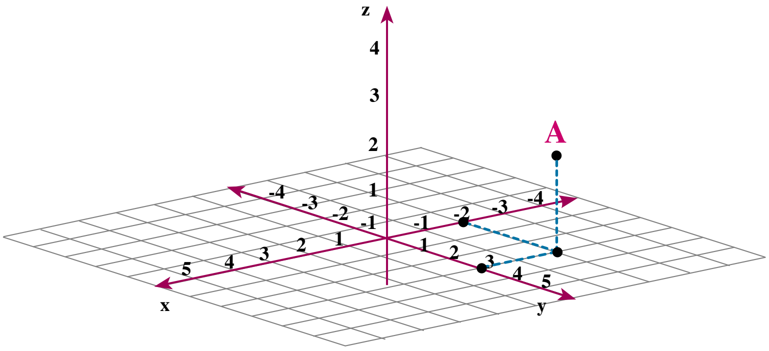


Example **(Identifying the position of a point in space)**

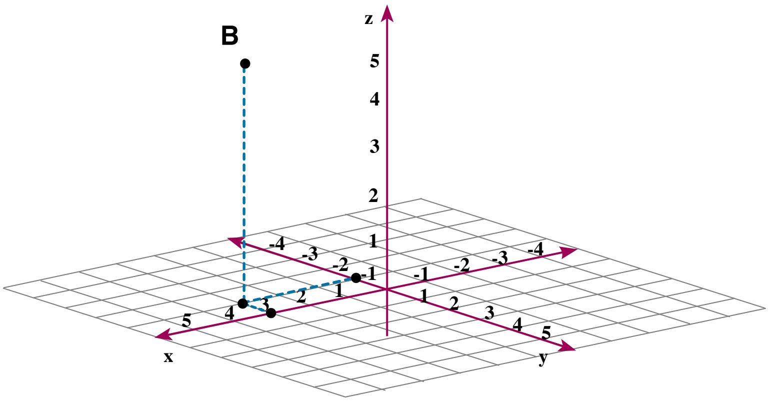
1 Identify the position of each of the following points using 3D-orthogonal coordinate system:

a A (-2, 3, 2) b B (3, -1, 5) c C (4, 0, -1)

Solution

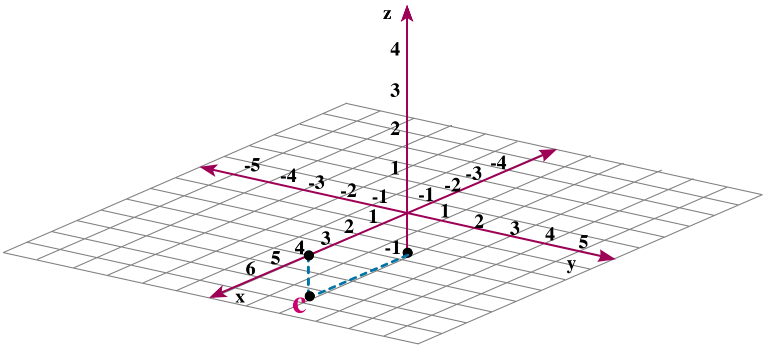


a To identify the point A (-2, 3, 2), we identify the point (-2, 3) in the x y plane, then move 2 units in the +ve direction of z-axis to get point A (-2, 3 , 2)



b To identify the point B (3, -1, 5), we identify the point (3, -1) in xy plane then move 5 units in the +ve direction of Z-axis to get point B.

c To identify the point C (4, 0, -1), we identify point (4, 0) on x-axis, then move one unit in the -ve direction of z-axis.



Try to solve

1 **a** Identify the position of each of the following points using 3-dimensional orthogonal coordinate system:

A (3, 2, 3) B (-1, 4, 3) C (0, 0, 4)

b **Complete:**

1- The distance between point A (-1, 2, 3) and the Cartesian xy-plane = ... unit length.

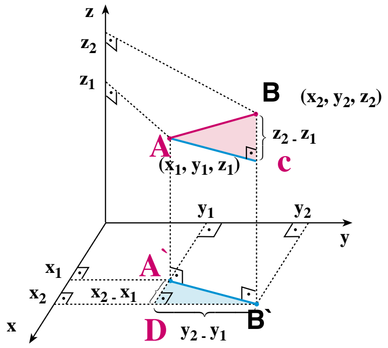
2- The distance between point B (4, -2, 1) and the cartesian yz-plane = ... unit length.

Learn

The distance between two points in space

If A (x_1 , y_1 , z_1), B (x_2 , y_2 , z_2) are two points in space, then the distance between A and B is given by the relation

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Example

2 Prove that the triangle ABC where A (2, -1, 3) , B(-4, 4, 2) and C (-2, 5, 1) is right angled at C.

Solution

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ **the distance rule**

$= \sqrt{(2 + 4)^2 + (-1 - 4)^2 + (3 - 2)^2} = \sqrt{62}$

$BC = \sqrt{(-4 + 2)^2 + (4 - 5)^2 + (2 - 1)^2} = \sqrt{6}$

$AC = \sqrt{(2 + 2)^2 + (-1 - 5)^2 + (3 - 1)^2} = \sqrt{56}$

$\because (AB)^2 = (\sqrt{62})^2 = 62$, $(BC)^2 + (AC)^2 = (\sqrt{6})^2 + (\sqrt{65})^2 = 6 + 56 = 62$

$\therefore (AB)^2 = (BC)^2 + (AC)^2$ $\therefore m(\hat{C}) = 90°$

Try to solve

2. Prove that the points A(4, 4, 0), B(4, 0, 4), C(0, 4, 4) are the vertices of an equilateral triangle and find its area.

Learn

The coordinates of a midpoint of a line segment

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ are two points in space, then the coordinates of point C which lies at the midpoint of $\overline{AB}$ is :

$$C = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example

3. If A(1, -3, 2), B (4, -1, 4), find the coordinates of the midpoint of $\overline{AB}$

Solution

The coordinates of the midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

$= \left(\frac{1+4}{2}, \frac{-3-1}{2}, \frac{2+4}{2}\right)$

$= \left(\frac{5}{2}, -2, 3\right)$

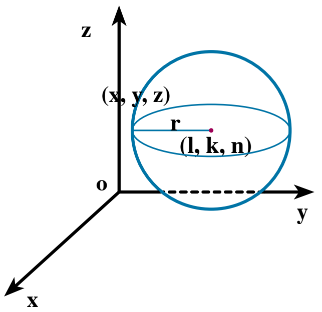
Try to solve

3. Find the coordinates of the midpoint of $\overline{CD}$ where C (0, 4, -2), D(-6, 3, 4)

Critical thinking: If C (2, 2, 6) is the midpoint of $\overline{AB}$ where A (1, -4 , 0), find coordinates of the point B

Learn

Equation of sphere in space



The sphere is identified by a set of points in space equidistant from a fixed point called (the center of the sphere) and the distance called (the radius length of the sphere).

If point (x, y, z) lies on the sphere of center (ℓ, k, n) and radius length r according to the rule of the distance between two points, then $r = \sqrt{(x - L)^2 + (y - k)^2 + (z - n)^2}$ by squaring both sides, we get the standard form of the equation of the sphere $(x - L)^2 + (y - k)^2 + (z - n)^2 = r^2$

Notice : The general equation of the sphere is:
$x^2 + y^2 + z^2 + 2Lx + 2ky + 2nz + d = 0$ which represents a sphere whose centre is $(-\ell, -k, -n)$ and radius length $r = \sqrt{\ell^2 + k^2 + n^2 - d}$ where $\ell^2 + k^2 + n^2 > d$

Example

4 Find the standard form of the equation of the sphere where center is (2, -1, 4)and radius length is 3 units.

Solution

the equation of the sphere is $(x - 2)^2 + (y + 1)^2 + (z - 4)^2 = 9$

Try to solve

4 Find the equation of the sphere whose center is the origin and its radius length is 5 units.

Example

5 Find the equation of the sphere in which A(-1, 5 ,4) , B(5, 1, -2) are the end points of its diameter.

Solution

The center of the sphere is the midpoint of $\overline{AB} = (\frac{-1+5}{2}, \frac{5+1}{2}, \frac{4-2}{2}) = (2, 3, 1)$

The radius length is the distance between the center and point A

$\therefore r = \sqrt{(2 + 1)^2 + (3 - 5)^2 + (1 - 4)^2} = \sqrt{22}$

$\therefore$ The equation of the sphere is: $(x - 2)^2 + (y - 3)^2 + (z - 1)^2 = 22$

Try to solve

5 Find equation of the sphere whose diameter is $\overline{AB}$ where A (-1, 4 , 2), B (3, -2, 6)

Example

6 Identify the center and the radius length of the sphere whose equation is
$x^2 + y^2 + z^2 + 4x - 2y - 6z + 11 = 0$

Solution

The coordinates of the center $= (-\frac{1}{2}$ coeff. of x, - coff. of y, $-\frac{1}{2}$coeff. of z$) = (-2, 1, 3)$
$\therefore r = \sqrt{\ell^2 + k^2 + n^2 - d} = \sqrt{(-2)^2 + (1)^2 + (3)^2 - 11} = \sqrt{3}$ unit length

Try to solve

6 Identify the center and the radius length of the sphere whose equation is
$x^2 + y^2 + z^2 + 6x - 8y + 4z + 1 = 0$

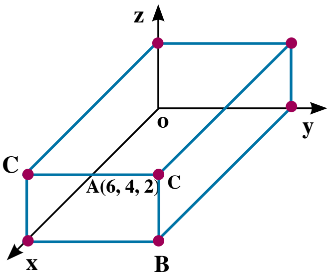
Exercises 1 - 1

Complete the following:

1. If point (x, y, z) lies in x y - plane, then z = ...

2. The two straight lines $\overleftrightarrow{xx'}$ and $\overleftrightarrow{zz'}$ form the coordinate plane .... whose equation is ....

3. The opposite figure represents a cuboid in an orthogonal coordinate system. One of it's vertices is the origion O(0, 0, 0) ,then



   the coordinates of the point B is ....

   Coordinates of the point C is ....

4. If A (1, -1, 4) , B (0, -3, 2), then the coordinates of the midpoint of $\overline{AB}$ is ....

5. The equation of the sphere of center (2, -1, 4) and radius length 5 units is ....

Choose the correct answer from the following:

6. The distance between point (3, -1, 2) and the Cartesian xz plane is .... unit length
   a) 3 b) -1 c) 2 d) 1

7. The length of the perpendicular drawn from point (-2, 3, 4) to the x-axis is .... unit length.
   a) 2 b) 3 c) 5 d) 4

8. The coordinates of the midpoint of the line segment whose terminals (-3, 2, 4), (5, 1, 8) is
   a) $(1, \frac{3}{2}, 6)$ b) (2, -1, 4) c) (8, -1, 4) d) $(1, -\frac{3}{2}, 2)$

9. The equation of the sphere whose center is the origin and radius length 5 units is
   a) $x^2 + y^2 + z^2 = 5$ b) $x^2 + y^2 + z^2 = 0$
   c) $(x - 5)^2 + (y - 5)^2 + (z - 5)^2 = 25$ d) $x^2 + y^2 + z^2 = 25$

10. The equation of a sphere with center (2, -3, 4) and touches xy-plane is
   a) $(x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 4$ b) $(x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 9$
   c) $(x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 16$ d) $(x + 2)^2 + (y - 3)^2 + (z + 4)^2 = 16$

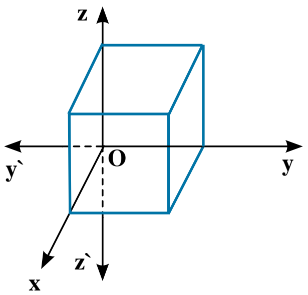
Answer the following questions :

11. Find the distance between the two points A and B in each of the following :
   a) A (7 , 0, 4) , B (1 , 0 , 0) b) A (4, 1, 9) , B (2 , 1 , 6)
   c) A (1 , 1 , -7) , B (-2 , -3 , -7)

12 Prove that the triangle whose vertices are the given points is a right angled triangle and find its Area:

a (-2 , 5 , 2), (0 , 0 , 2), (0 , 4 , 0) b (-4 , 4 , 1), (2 , -1 , 2), (-2 , 5 , 0)

13 The opposite figure represents a cube whose volume is 27 cubic unit and one of its vertices is the origin, find coordinates of the other vertices .



14 If the points (7, 1, 3) , (5, 3, k), (3, 5, 3) are the vertices of a triangle , prove that this triangle is isosceles, then find the value (values) of k which make (s) the triangle equilateral

15 Find the coordinates of the midpoint of $\overline{AB}$ in each of the following:

a A(3, -1, 4) , B (2, 0, -1) b A(-3, 5, 5) , B (-6, 4, 8)

16 If C (-1, 4, 0) is the midpoint of $\overline{AB}$ where B (4, -2, 1), find coordinates of point A.

17 Find the equation of the sphere if:

a the center is point (3, -1, 2) and its radius length $\sqrt{7}$

b (3, 4, -3), (0, 2, 1) are terminals of a diameter.

c the center is point (1, -6, 1) and passes through point (2, -1, 5)

18 Find the center and the radius length of the sphere in each of the following:

a $x^2 + y^2 + z^2 = 9$

b $x^2 + y^2 + z^2 - 2x + 4y = 0$

c $2x^2 + 2y^2 + 2z^2 - 2x - 6y - 4z + 5 = 0$

19 Find the equation of the sphere whose radius is 3 units and touches the Cartesian planes (given that three coordinates of the center are positive).

20 **Creative thinking:**

If A ∈ x -axis, B ∈ y -axis, C ∈ z-axis and if point(1, -1, 0) is the midpoint of $\overline{AB}$ and point (0, -1, 2) is the midpoint of $\overline{BC}$, find the coordinates of the midpoint of $\overline{AC}$

21 **Creative thinking:**

If x-axis cuts the sphere $(x - 2)^2 + (y + 3)^2 + (z - 1)^2 = 14$ at the two points A and B, find the length of $\overline{AB}$

22 **Writing in math:** If all the points in the space in the form of (x, y, 0) lie in x y-plane whose equation z = 0, find the equation of the plane in which all of its points in the form (x, y, 2)

23 **Discover the error:** If point B (-1, 4, 2) is the midpoint of $\overline{AC}$ where A(1 , 0 , 2), find C

Ashraf's solution	Zeiad's solution
$C = \left(\left(\frac{x_1 + x_2}{2}\right), \left(\frac{y_1 + x_2}{2}\right), \left(\frac{z_1 + z_2}{2}\right)\right)$ $= \left(\left(\frac{-1 + 1}{2}\right), \left(\frac{4 + 0}{2}\right), \left(\frac{2 + 2}{2}\right)\right)$ $= (0, 2, 2)$	let C (x, y, z) $\therefore \frac{1 + x}{2} = -1 \longrightarrow x = -3$ $\frac{0 + y}{2} = 4 \longrightarrow y = 8$ $\frac{2 + z}{2} = 2 \longrightarrow z = 2$ $\therefore$ c (-3, 8, 2)

Which answer is correct? Why?

Unit One

1 - 2 Vectors in space

You will learn

- Representing the vector by three components
- Position vector in space
- Fundamental unit vectors in space
- Express the vector interms t of he unit vectors
- Expressing the directed line segment in space in terms of its terminals
- Equality of two vectors in space
- The norm of a vector in space
- Unit vector in the direction of a vector in space
- Adding vector in space
- Multiply a vector by a real number

Key terms

- Position vector in space
- The norm vector
- Unit vector
- Scalar product
- Vector product

Introduction:

You have previously studied the scalar quantities and vector quantities and have known that the vector is represented by a directed segment determined by a magnitude [norm of the vector] and a direction. We will learn in this lesson the vectors in space (a-three dimensional coordinate system).

Learn

Position vector in space

The position vector of point A (A_x, A_y, A_z) with respect to the origin O (0, 0, 0) is defined as the directed line segment whose starting point is O and end point is A.

- The position vector of point A is denoted by $\vec{A}$ i.e. $\vec{A} = (A_x, A_y, A_z)$
- A_x is called the component of $\vec{A}$ in direction of x axis.
- A_y is called the component of $\vec{A}$ in direction of y axis.
- A_z is called the component of $\vec{A}$ in direction of z axis .

The norm of a vector

It is the length of the directed line segment which represents the vector.

If $\vec{A} = (A_x, A_y, A_z)$, then from the distance between two points rule

$$\|\vec{A}\| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$$

Example

1. If $\vec{A} = (2, -3, 1)$, $\vec{B} = (0, 4, -3)$ then

- The component of the vector $\vec{A}$ in the direction of x-axis is 2
- The component of the vector $\vec{B}$ in the direction of Z-axis is -3

$\|\vec{A}\| = \sqrt{2^2 + (-1)^2 + (3)^2} = \sqrt{14}$

$\|\vec{B}\| = \sqrt{(0)^2 + (4)^2 + (-3)^2} = 5$

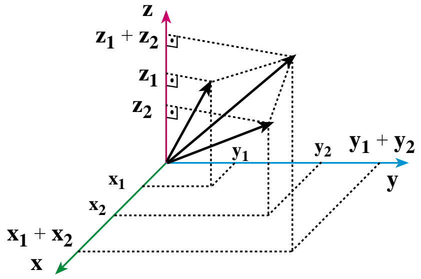
The vector $\vec{B}$ lies in the y z -plane (the component of the vector $\vec{A}$ vanishes in the direction of x-axis)

Try to solve

1) If $\overrightarrow{A} = (-1, 4, 2)$, $\overrightarrow{B} = (3, 1, 0)$, find

a) $A_x + B_y$ b) $\| \overrightarrow{A} \| + \| \overrightarrow{B} \|$

Adding Vectors in space



If $\overrightarrow{A} = (A_x\ A_y, A_z)$, $\overrightarrow{B} = (B_x, B_y, B_z)$, then:

$$\overrightarrow{C} = \overrightarrow{A} + \overrightarrow{B} = (A_x + B_x, A_y + B_y, A_z + B_z) = (C_x, C_y, C_z)$$

Example

2) If $\overrightarrow{A} = (-2, 3, 1)$, $\overrightarrow{B} = (0, -2, 4)$, then:

$$\overrightarrow{A} + \overrightarrow{B} = (-2, 3, 1) + (0, -2, 4) = (-2+0, 3+(-2), 1+4) = (-2, 1, 5)$$

Try to solve

2) If $\overrightarrow{A} = (4, -4, 0)$, $\overrightarrow{B} = (-1, 5, 2)$, find $\overrightarrow{A} + \overrightarrow{B}$

Properties of adding vectors in space

For any two vectors $\overrightarrow{A}$ and $\overrightarrow{B} \in R^3$, then:

1- **Closure property:** $\overrightarrow{A} + \overrightarrow{B} \in R^3$

2- **Commutative property:** $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{B} + \overrightarrow{A}$

3- **Associative property:** $(\overrightarrow{A} + \overrightarrow{B}) + \overrightarrow{C} = \overrightarrow{A} + (\overrightarrow{B} + \overrightarrow{C})$

4- **Identity element over addition (zero vector):** $\overrightarrow{O} = (0, 0, 0)$ is the neutral element of addition in R^3

i.e.: $\overrightarrow{A} + \overrightarrow{O} = \overrightarrow{O} + \overrightarrow{A} = \overrightarrow{A}$

5- **The additive inverse:** for every vector $\overrightarrow{A} = (A_x, A_y, A_z) \in R^3$ there is

$-\overrightarrow{A} = (-A_x, -A_y, -A_z) \in R^3$ such that: $\overrightarrow{A} + (-\overrightarrow{A}) = (-\overrightarrow{A}) + \overrightarrow{A} = \overrightarrow{O}$

Multiplying a vector by a scalar (a real number)

If $\overrightarrow{A} = (A_x, A_y, A_z) \in R^3$ and $K \in R$, then:

$$K\overrightarrow{A} = K(A_x, A_y, A_z) = (K A_x, KA_y, KA_z) \in R^3$$

For example:

$3\,(2, -1, 4) = (6, -3, 12)$

$\frac{1}{2}\,(4, 9, 6) = (2, \frac{9}{2}, 3)$

$-2\,(1, -3, -4) = (-2, 6, 8)$

Properties of multiplying vectors by a real numbers

If $\overrightarrow{A}$, $\overrightarrow{B} \in R^3$ and K, L $\in$ R, then

1. Distributive property

- $K(\overrightarrow{A} + \overrightarrow{B}) = K\overrightarrow{A} + K\overrightarrow{A}$
- $(K + L)\overrightarrow{A} = K\overrightarrow{A} + L\overrightarrow{A}$

2. Associative property

- $K(L\overrightarrow{A}) = L(K\overrightarrow{A}) = (KL)\overrightarrow{A}$

Example

3. If $\overrightarrow{A} = (-1, 5, 2)$, $\overrightarrow{B} = (4, -1, 3)$, then

1. $2\overrightarrow{A} - 3\overrightarrow{B} = 2(-1, 5, 2) - 3(4, -1, 3)$

$= (-2, 10, 4) + (-12, 3, -9)$

$= (-14, 13, -5)$

2. Find the vector $\overrightarrow{C}$ where $2\overrightarrow{C} + 3\overrightarrow{A} = 2\overrightarrow{B}$

$\because 2\overrightarrow{C} + 3\overrightarrow{A} = 2\overrightarrow{B}$ **by adding $-3\overrightarrow{A}$ for both sides**

$\therefore 2\overrightarrow{C} = 2\overrightarrow{B} - 3\overrightarrow{A}$

$\therefore 2\overrightarrow{C} = 2(4, -1, 3) - 3(-1, 5, 2)$

$= (8, -2, 6) + (3, -15, -6)$

$= (11, -17, 0)$ **by multiplying by $\frac{1}{2}$**

$\therefore \overrightarrow{C} = \frac{1}{2}(11, -17, 0) = (\frac{11}{2}, \frac{-17}{2}, 0)$

Try to solve

3. If $\overrightarrow{C} = (2, -3, 1)$, $\overrightarrow{D} = (0, 2, -2)$

a find $5\overrightarrow{C} - 2\overrightarrow{D}$

b If $3\overrightarrow{A} - 4\overrightarrow{D} = \overrightarrow{C}$, find $\overleftarrow{A}$

Equality of vectors in the space

If $\overrightarrow{A} = (A_x, A_y, A_z)$, $\overrightarrow{B} = (B_x, B_y, B_z)$, then:

$$\overrightarrow{A} = \overrightarrow{B} \text{ if and only if: } A_x = B_x , \quad A_y = B_y , \quad A_z = B_z$$

Example

(4) Find the values of L, m , n which make the two vectors $\overrightarrow{A} = (L - 4, m^2 - 3 , 1)$, $\overrightarrow{B} = (5, 1, n^2)$ equal

Solution

$\because \quad \overrightarrow{A} = \overrightarrow{B}$

$\therefore \quad A_x = B_x \longrightarrow L - 4 = 5 \longrightarrow L = 9$

$A_y = B_y \longrightarrow m^2 - 3 = 1 \longrightarrow m^2 = 4 \longrightarrow m = \pm 2$

$A_z = B_z \longrightarrow n^2 = 1 \longrightarrow n = \pm 1$

Try to solve

(4) If $(2x + 1 , 5 , k + 4) = (-1 , y^2 - 4 , x + 1)$, find the value of x, y and k?

The unit vector

The unit vector is a vector whose norm equals the unit length

For example :

$\overrightarrow{A} = (\frac{-3}{13}, \frac{4}{13}, \frac{12}{13})$ **is a unit vector because:** $\|\overrightarrow{A}\| = \sqrt{(\frac{-3}{13})^2 (\frac{4}{13})^2 + (\frac{12}{13})^2} = 1$

Try to solve

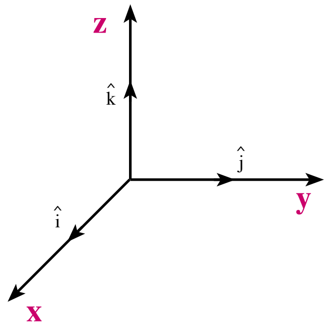
(5) Show which of the following vectors represents a unit vector

$\overrightarrow{A} = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ $\qquad \overrightarrow{B} = (\frac{1}{5}, \frac{4}{5}, \frac{-\sqrt{5}}{5})$

Fundamental unit vectors ($\hat{i}$, $\hat{j}$, $\hat{k}$)

It is a directed segments whose starting point is the origin point and its norm is the unit length and its direction is the positive direction of x, y and z axes respectively:

$\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$



Critical thinking

Express the vectors (-1, 0, 0) , (0, -1, 0), (0, 0, -1) interms of the fundamental unit vectors.

Expressing a vector in space in terms of the fundmental unit vectors

If $\overrightarrow{A} = (A_x, A_y, A_z) \in R^3$, then the vector $\overrightarrow{A}$ can be written in the form

$$\overrightarrow{A} = (A_x, 0, 0) + (0, A_y, 0) + (0, 0, A_z)$$
$$= A_x(1, 0, 0) + A_y(0, 1, 0) + A_z(0, 0, 1)$$
$$= A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

Example

5 If $\overrightarrow{A} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\overrightarrow{B} = -\hat{i} - 2\hat{j}$, find

a $2\overrightarrow{A} - 3\overrightarrow{B}$

b $\|\overrightarrow{A} + \overrightarrow{B}\|$, $\|\overrightarrow{A}\| + \|\overrightarrow{B}\|$ What do you notice?

Solution

a $2\overrightarrow{A} - 3\overrightarrow{B} = 2(2\hat{i} - 3\hat{j} + \hat{k}) - 3(-\hat{i} - 2\hat{j})$

$= 4\hat{i} - 6\hat{j} + 2\hat{k} + 3\hat{i} + 6\hat{j}$

$\therefore 2\overrightarrow{A} - 3\overrightarrow{B} = 7\hat{i} + 2\hat{k}$

b $\overrightarrow{A} + \overrightarrow{B} = (2\hat{i} - 3\hat{j} + \hat{k}) + (-\hat{i} - 2\hat{j})$

$= \hat{i} - 5\hat{j} + \hat{k}$

$\therefore \|\overrightarrow{A} + \overrightarrow{B}\| = \sqrt{1^2 + (-5)^2 + 1^2} = \sqrt{27}$

$\|\overrightarrow{A}\| + \|\overrightarrow{B}\| = \sqrt{2^2 + (-3)^2 + 1^2} + \sqrt{(-1)^2 + (-2)^2}$

$= \sqrt{14} + \sqrt{5}$ **we notice that** $\|\overrightarrow{A} + \overrightarrow{B}\| \neq \|\overrightarrow{A}\| + \|\overrightarrow{B}\|$

Try to solve

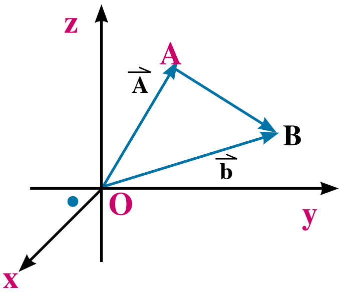
6 If $\overrightarrow{A} = -3\hat{j} - \hat{k} + 5\hat{i}$, $\overrightarrow{B} = -2\hat{k} + 3\hat{i}$, find

a $3\overrightarrow{A} - 5\overrightarrow{B}$

b $\|\overrightarrow{A} - \overrightarrow{B}\|$

Expressing a directed line segment in the space in terms of the coordinates of its terminals

let A and B be two points in space and their position vectors with respect to the origin be $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.



$\because \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

or $\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$

Example

6 If A (-2, 3, 1) , B (4, 0, 2) , then

$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$

$= (4, 0, 2) - (-2, 3, 1) = (6, -3, 1)$

$\overrightarrow{BA} = \overrightarrow{A} - \overrightarrow{B}$

$= (-2, 3, 1) - (4, 0, 2) = (-6, 3, -1)$

Notice that: $\overrightarrow{AB} = -\overrightarrow{BA}$

Try to solve

7 a If A (2, -3, 0), B (1, 4, -1) , find $\overrightarrow{AB}$

b If A (1, 1, -2), $\overrightarrow{AB}$ (4, -1, 2) , find coordinates of point B

The unit vector in a direction of a given vector

If $\overrightarrow{A} = (A_x, A_y, A_z) \in R^3$, then the unit vector in the direction of the vector $\overrightarrow{A}$ is denoted by $\overrightarrow{U_A}$ and given by the relation :

$$\overrightarrow{U_A} = \frac{\overrightarrow{A}}{\| \overrightarrow{A} \|}$$

Example

7 If $\overrightarrow{A} = (-2, 2, 1)$, $\overrightarrow{B} = (3, 1, -2)$, find the unit vector in the direction of $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{AB}$

Solution

$$\overrightarrow{U_A} = \frac{\overrightarrow{A}}{\| \overrightarrow{A} \|} = \frac{(-2, 2, 1)}{\sqrt{4+4+1}} = \left(\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$\overrightarrow{U_B} = \frac{\overrightarrow{B}}{\| \overrightarrow{B} \|} = \frac{(3, 1, -2)}{\sqrt{9+1+4}} = \left(\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}\right)$$

$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$

$= (3, 1, -2) - (-2, 2, 1) = (5, -1, -3)$

$$\therefore \overrightarrow{U_{AB}} = \frac{\overrightarrow{AB}}{\| \overrightarrow{AB} \|}$$

$$= \frac{(5, -1, -3)}{\sqrt{25+1+9}} = \left(\frac{5}{\sqrt{35}}, \frac{-1}{\sqrt{35}}, \frac{-3}{\sqrt{35}}\right)$$

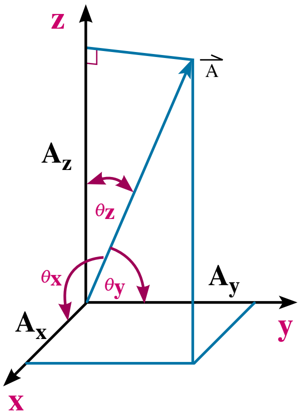
Try to solve

8 Find the unit vector in the direction of each of the following vectors:

a $\overrightarrow{A} = (8, -4, -8)$ b $\overrightarrow{B} = \hat{i} - 2\hat{j} - \hat{k}$ c $\overrightarrow{C} = 3\hat{i} - 4\hat{k}$

Direction angles and direction cosines of a vector in space

If $\overrightarrow{A} = (A_x, A_y, A_z)$ is a vector in space and $(\theta_x, \theta_y, \theta_z)$ are the measures of the angles made by the vector with the +ve directions of x, y, z axes respectively, then:



$A_x = \|\overrightarrow{A}\| \cos\theta_x$, $A_y = \|\overrightarrow{A}\| \cos\theta_y$, $A_z = \|\overrightarrow{A}\| \cos\theta_z$

$\therefore \overrightarrow{A} = \|\overrightarrow{A}\| \cos\theta_x \hat{i} + \|\overrightarrow{A}\| \cos\theta_y \hat{j} + \|\overrightarrow{A}\| \cos\theta_z \hat{k}$

$= \|\overrightarrow{A}\| (\cos\theta_x \hat{i} + \cos\theta_y \hat{j} + \cos\theta_z \hat{k})$

$(\theta_x, \theta_y, \theta_z)$ **are called the direction angles of vector** $\overrightarrow{A}$

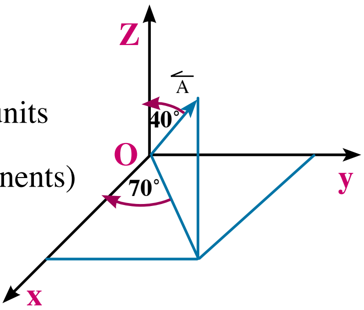
$\cos\theta_x, \cos\theta_y, \cos\theta_z$ **are called the direction cosines of vector** $\overrightarrow{A}$

Notice that: $\cos\theta_x \hat{i} + \cos\theta_y \hat{j} + \cos\theta_z \hat{k}$ represent the unit vector in the direction of the vector $\overrightarrow{A}$ i.e.

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

Example

8 The opposite figure represents a vector $\overrightarrow{A}$ whose norm is 10 units



a Express the vector $\overrightarrow{A}$ in algebraic form (Cartesian components)

b Find the measure of the direction angles of $\overrightarrow{A}$

Solution

First resolve $\overrightarrow{A}$ into two components; the first in the direction of $\overrightarrow{OZ}$ with magnitude

$A_z = \|\overrightarrow{A}\| \cos\theta_z = 10 \cos 40 = 7.66$

The second in xy-plane

$A_{xy} = \|\overrightarrow{A}\| \sin\theta_z = 10 \sin 40 = 6.428$

Now, resolve the component A_{xy} into two components; the first is in the direction of $\overrightarrow{OX}$ with magnitude

$A_x = A_{xy} \cos 70 = 6.428 \cos 70 = 2.199$

the second is in the direction of $\overrightarrow{OY}$ with magnitude

$A_y = A_{xy} \sin 70 = 6.428 \sin 70 = 6.04$

So, the Cartesian form of the vector $\overrightarrow{A}$ is

$$\overrightarrow{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$$
$$= 2.199\,\hat{i} + 6.04\,\hat{j} + 7.66\,\hat{k}$$

Second : to find the measures of the direction angles, we get the unit vector in the direction of $\overrightarrow{A}$

$$\overleftarrow{U}_A = \frac{\overrightarrow{A}}{\|\overrightarrow{A}\|} = \frac{1}{10}(2.199\,\hat{i} + 6{,}04\,\hat{j} + 7{,}66\,\hat{k})$$
$$= 0.2199\,\hat{i} + 0{,}604\,\hat{j} + 0{,}766\,\hat{k}$$

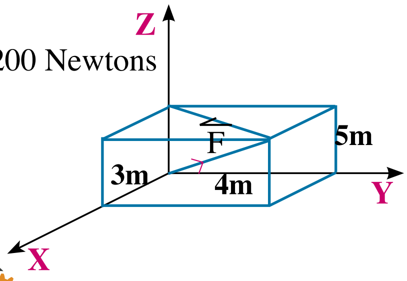
$\therefore \cos\theta_x = 0.2199$, **then** $\theta_x = \cos^{-1}(0.2199) = 77.3°$

$\cos\theta_y = 0.604$, **then** $\theta_y = \cos^{-1}(0.604) = 52.84°$

$\cos\theta_z = 0.766$, **then** $\theta_z = \cos^{-1}(0.766) = 40°$

Try to solve

9. The opposite figure represents the force $\overrightarrow{F}$ with a magnitude 200 Newtons
   - a) Express the force $\overrightarrow{F}$ in an algebraic form.
   - b) Find the measures of the direction angles of the force $\overrightarrow{F}$.



Exercises 1 - 2

Complete the following:

1. If $\overrightarrow{A} = (-3, 4, 2)$, then $\|\overrightarrow{A}\| =$ ....

2. If $\overrightarrow{A} = \hat{i} - 2\,\hat{j} + 3\,\hat{k}$, $\overrightarrow{B} = 3\,\hat{i} - \hat{k}$, then $\overrightarrow{A} - \overrightarrow{B} =$ ....

3. The unit vector in the direction of $\overrightarrow{AB}$ where A (-1, 2, 0) , B (3, -1, 2) is ....

4. The vector $\overrightarrow{A} = 3\,\hat{i} + \hat{j} - 2\,\hat{k}$ makes an angle of measure .... with the +ve direction of x-axis.

5. The vector $\overrightarrow{B} = \hat{i} + 2\,\hat{j}$ makes an angle of measure .... with the +ve direction of z-axis.

Choose the correct answer from the following:

6. If $\overrightarrow{A} = (-2, k, 1)$ and $\|\overrightarrow{A}\| = 3$ unit, then k = ...
   - a) 4
   - b) -4
   - c) ± 2
   - d) $\sqrt{14}$

7. If 30°, 70°, θ are the direction angles of a vector, then one of the values of θ = ...

a) 100° b) 80° c) 260° d) 68,61°

8. If $\overrightarrow{A}$ = (-1, 5, -2), $\overrightarrow{B}$ = (3, 1, 1) and if $\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C} = \hat{i}$, then $\overrightarrow{C}$ = ...

a) $\hat{i} + 6\hat{j} - \hat{k}$ b) $-\hat{i} - 6\hat{j} + \hat{k}$

c) $\hat{i} + 4\hat{j} - 3\hat{k}$ d) $\hat{i} + 4\hat{j} - \hat{k}$

9. The direction cosines of the vector $\overrightarrow{A}$ = (-2, 1, 2) is ....

a) (-2, 1, 2) b) $(\frac{-2}{3}, \frac{1}{3}, \frac{2}{3})$ c) $(\frac{5}{-2}, 5, \frac{5}{2})$ d) (-1, 1, 1)

Answer the following :

10. If $\overrightarrow{A}$ = (2, -3, 1) , $\overrightarrow{B}$ = (4, -2, 0) , $\overrightarrow{C}$ = (-6, 0, 3), find each of the following vectors:

a) $\overrightarrow{A} + \overrightarrow{B}$ b) $3\overrightarrow{A} - \frac{1}{3}\overrightarrow{C}$ c) $\frac{3}{2}\overrightarrow{B} + \frac{2}{3}\overrightarrow{C}$

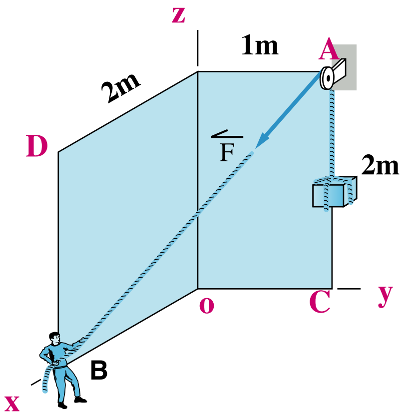
11. If $\overrightarrow{A} = 2\hat{i} - 3\hat{j} + 5\hat{k}$, $\overrightarrow{B} = 4\hat{j} - 2\hat{k}$, $\overrightarrow{C} = 4\hat{i} + 5\hat{j} - 6\hat{k}$, find each of the following vectors :

a) $2\overrightarrow{A} + \overrightarrow{B}$ b) $\frac{1}{2}\overrightarrow{B} - \overrightarrow{C}$ c) $3\overrightarrow{A} - 2\overrightarrow{C}$

12. Find the norm of each of the following vectors:

a) $\overrightarrow{A}$ = (2, -1, 0) b) $\overrightarrow{B}$ = (1, 2, -2) c) $\overrightarrow{C} = \hat{j}$ d) $\overrightarrow{D} = \hat{i} - 4\hat{j}$

13. If $\overrightarrow{A}$ = (k, 0, 0) , $\hat{i}$ = (1, 0, 0), prove that $\| \overrightarrow{A} \| = |k| \, \| \hat{i} \|$



14. If the tension force in a stringe equals 21 newtons , fine the algebraic components of the force $\overrightarrow{F}$ in the directions of the Cartesian axes

15. **Open question:** What can you say about the coordinates of the vector $\overrightarrow{A}$ if the vector $\overrightarrow{A}$ is parallel to y z-plane ?

16. **Open question:** If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors in R^3. Is $\| \overrightarrow{A} + \overrightarrow{B} \| = \| \overrightarrow{A} \| + \| \overrightarrow{B} \|$? Which side is greater if both sides are unequal ?

17. **Creative thinking:** Find the Algebric form of the vector $\overrightarrow{A}$ if its norm is 5 units and makes equal angles with the +ve directions of the Cartesian axes .

Unit One

1 - 3 Vectors multiplication

You will learn

- Scalar product of two vectors in a plane and space
- Parallel and perpendicular vectors.
- The angle between two vectors
- The component of a vector in the direction of another vector
- The projection of a vector in the direction of another vector
- The work done by a force
- The cross product of two vectors in a plane and space
- The geometrical meaning of the cross product
- The right hand system of unit vectors
- The scaler triple product
- The geometrical meaning of scalar triple product

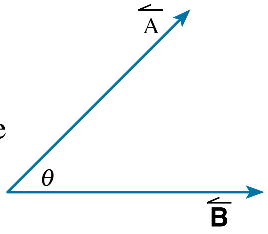
Key terms

- Scalar product
- vector product
- Component
- Unit vector
- Work
- Right hand rule
- Scalar triple product

You have learned earlier how to make some operations on the vectors such as adding and multiplying a vector by scaler but you might ask yourself. Is it possible to multiply vectors? The answer is yes. There are two types of multiplying vectors; scalar product and vector product between two vectors. In this lesson, we will study the two types and their Algebraic and geometrical properties and their physical Applications to help you study mechanics.

Scalar product of two vectors (Dot product)

Think and discuss

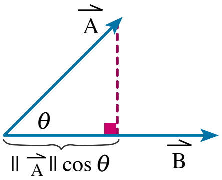


If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors and θ is the measure angle between them, find:

1- the component of $\overrightarrow{A}$ in the direction of $\overrightarrow{B}$.

2- the product of the norm of vector $\overrightarrow{B}$ and the component of $\overrightarrow{A}$ in the direction of vector $\overrightarrow{B}$.

From Think and discuss, we conclude that:



1- The component of the vector $\overrightarrow{A}$ in the direction of vector $\overrightarrow{B}$ $= \| \overrightarrow{A} \| \cos\theta$

2- The product of the norm of vector $\overrightarrow{B}$ and the component of vector $\overrightarrow{A}$ in the direction of vector $\overrightarrow{B}$ equals $\| \overrightarrow{B} \| \, \| \overrightarrow{A} \| \cos\theta$

The absolute value of this quantity represents the area of a rectangle whose dimensions are the norm of the vector $\overrightarrow{B}$ and the component of vector $\overrightarrow{A}$ in the direction of vector $\overrightarrow{B}$.

Learn

The scalar product of two vectors (Dot product)

If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors and the measure of the angle between them is θ, then the area of the rectangle whose dimensions are the norm of one of them and the component of the other vector on it is known as the scaler product of the two vectors denoted by $\overrightarrow{A} \cdot \overrightarrow{B}$

i.e. $\vec{A} \cdot \vec{B} = \| \vec{A} \| \, \| \vec{B} \| \cos \theta$

Example

1 If $\vec{A}$ and $\vec{B}$ two vectors of the angle between them 60° and $\| \vec{A} \| = 2 \| \vec{B} \| = 8$, find $\vec{A} \cdot \vec{B}$

Solution

$\| \vec{A} \| = 2 \| \vec{B} \| = 8 \longrightarrow \| \vec{A} \| = 8 , \| \vec{B} \| = 4$

from the definition of the scaler product

$\vec{A} \cdot \vec{B} = \| \vec{A} \| \, \| \vec{B} \| \cos \theta$

$= 8 \times 4 \times \cos 60^\circ = 16$

Try to solve

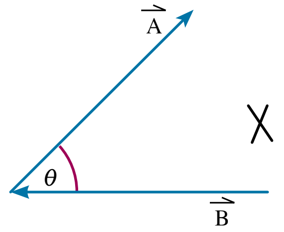
1 If $\vec{A}$ and $\vec{B}$ are two vectors and the measure of the angle between them is 135° , $\| \vec{A} \| = 6$, $\| \vec{A} \| = 10$, find $\vec{A} \cdot \vec{B}$

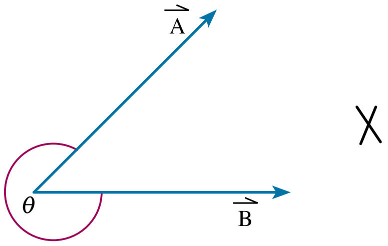
Critical thinking: What are the cases at which the value of the scaler product is equal to zero?

Important notes

1- To identify the angle between two vectors, the two vectors must be both getting in or out of the common point.

2- The measure of the angle between the two vectors $\in [0 , \pi]$





Example

2 If $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors of a right hand system, find $\hat{i} \cdot \hat{i}$, $\hat{j} \cdot \hat{j}$, $\hat{k} \cdot \hat{k}$

Solution

$\hat{i} \cdot \hat{i} = \| \hat{i} \| \, \| \hat{i} \| \cos 0^\circ$

$= 1 \times 1 \times 1 = 1$

similar $\hat{j} \cdot \hat{j} = 1$, $\hat{k} \cdot \hat{k} = 1$

Remember

The norm of the unit vector equals one.

Try to solve

2 If $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors of a right hand system, find $\hat{i} \cdot \hat{j}$, $\hat{j} \cdot \hat{k}$, $\hat{k} \cdot \hat{i}$

Properties of the scaler product

From the previous example, we can conclude the properties of the scaler product:

1- $\overrightarrow{A} \cdot \overrightarrow{B} = \overrightarrow{B} \cdot \overrightarrow{A}$ — **Commutative property**

2- $\overrightarrow{A} \cdot \overrightarrow{A} = \|\overrightarrow{A}\|^2$

3- $\overrightarrow{A} \cdot \overrightarrow{B} = 0$ — **If and only if $\overrightarrow{A}$, $\overrightarrow{B}$ are perpendicular (condition of perpendicularity)**

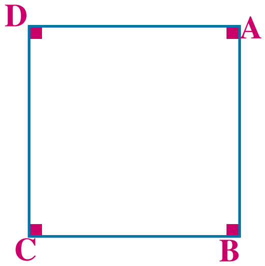
4- $\overrightarrow{A} \cdot (\overrightarrow{B} + \overrightarrow{C}) = \overrightarrow{A} \cdot \overrightarrow{B} + \overrightarrow{A} \cdot \overrightarrow{C}$ — **distributive property**

5- $(k\overrightarrow{A}) \cdot \overrightarrow{B} = \overrightarrow{A} \cdot (k\overrightarrow{B}) = k(\overrightarrow{A} \cdot \overrightarrow{B})$ — **K is a real number**

Example

(3) ABCD is a square with side length 10cm, find

(a) $\overrightarrow{AB} \cdot \overrightarrow{DC}$ (b) $\overrightarrow{AB} \cdot \overrightarrow{BC}$ (c) $\overrightarrow{AB} \cdot \overrightarrow{CA}$



Solution

(a) $\because$ $\overrightarrow{AB}$, $\overrightarrow{DC}$ are parallel and on the same direction

$\therefore$ the angle between them is = zero°

$\therefore \overrightarrow{AB} \cdot \overrightarrow{DC} = \|\overrightarrow{AB}\| \, \|\overrightarrow{DC}\| \cos 0°$

$= 10 \times 10 \times 1 = 100$

(b) $\overrightarrow{AB}$, $\overrightarrow{BC}$ are perpendicular **the measure of the angle between them is 90°**

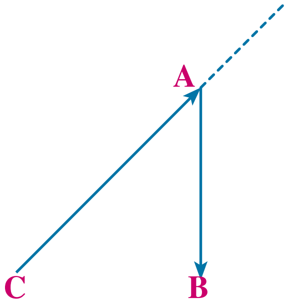
$\therefore \overrightarrow{AB} \cdot \overrightarrow{BC}$ = zero

(c) $\because$ $\overrightarrow{AB}$, $\overrightarrow{CA}$ **do not start at the same point**

$\therefore$ extend $\overrightarrow{CA}$ so the measure of the angle between them is 135°

$\overrightarrow{AB} \cdot \overrightarrow{CA} = \|\overrightarrow{AB}\| \, \|\overrightarrow{CA}\| \cos 135^o$

$= 10 \times 10\sqrt{2} \times \frac{-1}{\sqrt{2}} = -100$



Another solution (c)

$\overrightarrow{AB} \cdot \overrightarrow{CA} = \overrightarrow{AB} \cdot (-\overrightarrow{AC})$

$= -\overrightarrow{AB} \cdot \overrightarrow{AC}$

$= -\|\overrightarrow{AB}\| \, \|\overrightarrow{CA}\| \cos 45$

$= -10 \times 10\sqrt{2} \times \frac{1}{\sqrt{2}} = -100$

Try to solve

(3) ABC is an equilateral triangle with side length 8 cm, find each of:

(a) $\overrightarrow{AB} \cdot \overrightarrow{AC}$ (b) $\overrightarrow{AB} \cdot \overrightarrow{BC}$ (c) $(2\overrightarrow{AC}) \cdot (3\overrightarrow{CB})$

Learn

The scalar product of two vectors in an orthogonal coordindte system

If $\overrightarrow{A} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}), B = (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$ then

$\overrightarrow{A} \cdot \overrightarrow{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$ using the distribution property

$$= A_x B_x \hat{i} \cdot \hat{i} + A_x B_y \hat{i} \cdot \hat{j} + A_x B_z \hat{i} \cdot \hat{k}$$
$$+ A_y B_x \hat{j} \cdot \hat{i} + A_y B_y \hat{j} \cdot \hat{j} + A_y B_z \hat{j} \cdot \hat{k}$$
$$+ A_z B_x \hat{k} \cdot \hat{i} + A_z B_y \hat{k} \cdot \hat{j} + A_z B_z \hat{k} \cdot \hat{k}$$

$\because \quad \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

$\therefore \overrightarrow{A} \cdot \overrightarrow{B} = A_x B_x + A_y B_y + A_z B_z$

Note

If $\overrightarrow{A} = (A_x , A_y)$, $\overrightarrow{B} = (B_x , B_y)$ in the coordinates plane,
then $\overrightarrow{A} \cdot \overrightarrow{B} = A_xB_x + A_yB_y$

Example

4 If $\overrightarrow{A} = 2 \hat{i} + 3 \hat{j} + 4 \hat{k}$, $\overrightarrow{B} = - \hat{i} - 2 \hat{j} + \hat{k}$, find $\overrightarrow{A} \cdot \overrightarrow{B}$

Solution

$\overrightarrow{A} \cdot \overrightarrow{B} = (2, 3, 4) \cdot (-1, -2, 1)$
$= 2 \times -1 + 3 \times (-2) + 4 \times 1$
$= -2 - 6 + 4 = -4$

Try to solve

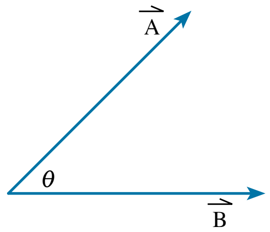
4 Find $\overrightarrow{A} \cdot \overrightarrow{B}$ in each of the following:

a $\overrightarrow{A} = (-1, 3, 2)$, $\overrightarrow{B} = 4 \hat{i} - 2 \hat{j} + 5 \hat{k}$ What do you deduce?

b $\overrightarrow{A} = 2 \hat{i} - \hat{j}$, $\overrightarrow{B} = \hat{j} - 3 \hat{i}$

Learn

The angle between two vectors



You know that $\overrightarrow{A} \bullet \overrightarrow{B} = \| \overrightarrow{A} \| \, \| \overrightarrow{B} \| \cos \theta$

where θ is the measure of angle between two non zero vectors

$\overrightarrow{A}$, $\overrightarrow{B}$, $0^\circ \leqslant \theta \leqslant 180^\circ$

$$\therefore \cos \theta = \frac{\overrightarrow{A} \bullet \overrightarrow{B}}{\| \overrightarrow{A} \| \, \| \overrightarrow{B} \|}$$

Speical cases:

1- **If** $\cos \theta = 1$, **then** $\overrightarrow{A}$, $\overrightarrow{B}$ are parallel and in the same direction.

2- **If** $\cos \theta = -1$, **then** $\overrightarrow{A}$, $\overrightarrow{B}$ are parallel and in the opposite direction.

3- **If** $\cos \theta = 0$, **then** $\overrightarrow{A}$, $\overrightarrow{B}$ are perpendicular.

Example

(5) Find measure of the angle between the two vectors
$\overrightarrow{A} = 4\,\hat{i} + 3\,\hat{j} + 7\,\hat{k}$, $\overrightarrow{B} = 2\,\hat{i} + 5\,\hat{j} + 4\,\hat{k}$.

Solution

$\| \overrightarrow{A} \| = \sqrt{4^2 + 3^2 + 7^2} = \sqrt{74}$

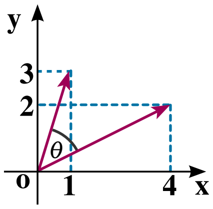
$\| \overrightarrow{B} \| = \sqrt{2^2 + 5^2 + 4^2} = \sqrt{45}$

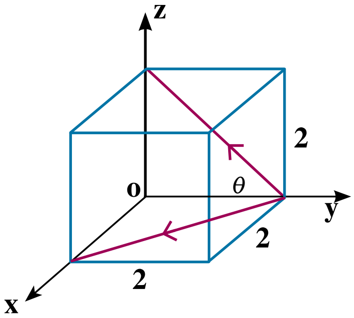
$$\cos \theta = \frac{\overrightarrow{A} \bullet \overrightarrow{B}}{\| \overrightarrow{A} \| \, \| \overrightarrow{B} \|} = \frac{(4, 3, 7) \bullet (2, 5, 4)}{\sqrt{74}\,\sqrt{45}} = \frac{8 + 15 + 28}{\sqrt{74}\,\sqrt{45}} = \frac{51}{\sqrt{74}\,\sqrt{45}}$$

$$\therefore \cos^{-1} \left(\frac{51}{\sqrt{74}\,\sqrt{45}}\right) = \cos^{-1} (0.8838) = 27.9^\circ$$

Try to solve

(5) Find θ in each of the following:



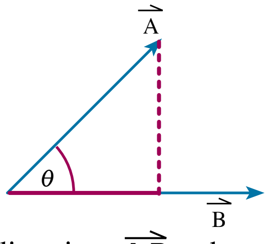


Learn

The componont (projection) of a vector in the direction of another vector.

If $\vec{A}$ and $\vec{B}$ are vectors, **then** the component of vector $\vec{A}$ is in the direction $\vec{B}$ (denoted A_B) is

$$A_B = \|\vec{A}\| \cos\theta = \frac{\vec{A} \cdot \vec{B}}{\|\vec{B}\|}$$



Example

6 Find the component of the force $\vec{F} = 2\hat{i} - 3\hat{j} + 5\hat{k}$ in the direction $\vec{AB}$ where A(1, 4, 0), B(-1, 2, 3)

Solution

$\vec{AB} = \vec{B} - \vec{A}$

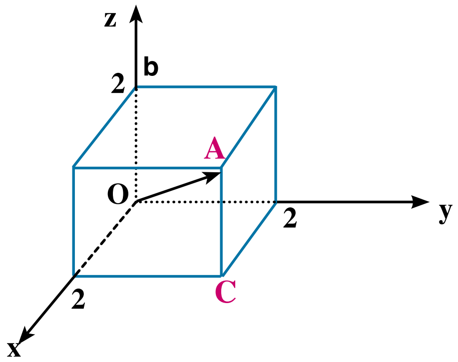
$= (-1, 2, 3) - (1, 4, 0) = (-2, -2, 3)$

The component of the force $\vec{F}$ in the direction of $\vec{AB} = \dfrac{\vec{F} \cdot \vec{AB}}{\|\vec{AB}\|}$

$$= \frac{(2, -3, 5) \cdot (-2, -2, 3)}{\sqrt{(-2)^2 + (-2)^2 + {}^2 3}} = \frac{17}{\sqrt{17}} = \sqrt{17}$$

Try to solve

6 The opposite figure represents a cube with side length 2 length unit, find the projection of $\vec{OA}$ on the vector $\vec{CB}$



Critical thinking: when does the component of a vector in a direction of another vector vanish?

Learn

Using scalar product to find the work done by a force

If a force $\vec{F}$ acts on a body to move it a displacement $\vec{S}$, we say that the force does work which can be found by the relation:

$$W = \vec{F} \cdot \vec{S} = \|\vec{F}\| \, \|\vec{S}\| \cos\theta$$

The unit of measuring the work is the unit of measuring the force × unit of measuring the displacement.

Example

⑦ A force $\overrightarrow{F} = \hat{i} - 2\hat{j} + 3\hat{k}$ newtons acts on a body to move it from point A (-3, 1, 0) to point B(2, 0, -2). Find the work done by the force $\overrightarrow{F}$ where the displacement is measured in meter.

Solution

$\overrightarrow{S} = \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$

$= (2, 0, -2) - (-3, 1, 0)$

$= (5, -1, -2)$

$W = \overrightarrow{F} \cdot \overrightarrow{S}$

$= (1, -2, 3) \cdot (5, -1, -2) = 1$ N.m (joule)

Tip

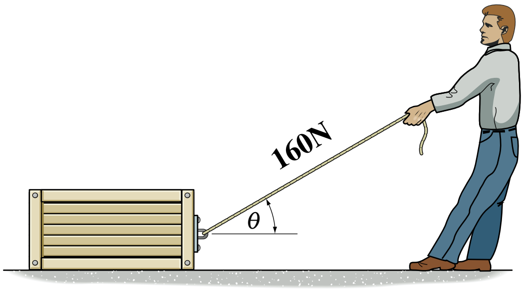
If the force is measured by "Newton" and the dicplacement is measured by "metre", then the work is measured by "joule"

Try to solve

⑦ A body moves under the action of a force $\overrightarrow{F} = -6\hat{i} + 8\hat{j}$ from point A (-1, 3) to point B (4, 7), find the work done by the force .

Example

⑧ **In the opposite figure :** A man pulls a box with a 160N tension force inclined to the horizontal with on angle whose tangent $\frac{3}{4}$ to move it a horizontal distance of magnitude 5 m, find the work done by the tension force.



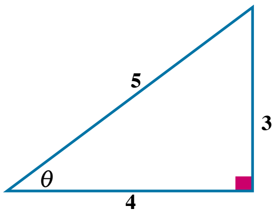
Solution

Work $= \overrightarrow{F} \cdot \overrightarrow{S}$

$= \| \overrightarrow{F} \| \, \| \overrightarrow{S} \| \cos\theta$

$= 160 \times 5 \times \frac{4}{5}$

$= 640$ joule



Try to solve

⑧ **In the opposite figure:** A person lifts A box using a string passing over a smooth pully and inclines to the vertical with an angle of measure 30º. If the tension force in the string is 120 newten to rise the box 3meter above the ground, find the worked done by the tension force.

Learn

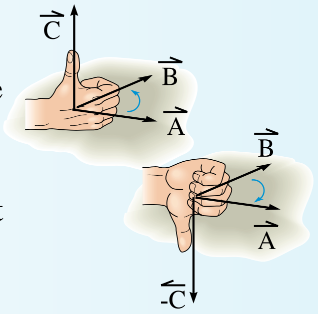
Vector product of two vectors (cross product)

If $\vec{A}$ and $\vec{B}$ are two vectors in a plane enclosing an angle of measure θ and $\vec{C}$ is a unit vector perpendicular to the plane which contains $\vec{A}$ and $\vec{B}$, **then** the cross product of $\vec{A}$ and $\vec{B}$ is given by the relation

$$\vec{A} \times \vec{B} = (\| \vec{A} \| \| \vec{B} \| \sin \theta) \vec{C}$$

The direction of the unit vector $\vec{C}$ is defined (up or down) According to the right hand rule where the curved fingers of the right hand show the direction of the relation from the vector $\vec{A}$ to the vector $\vec{B}$, then the thump shows the direction of the vector $\vec{C}$

Important notes



1- **If** $\vec{A} \times \vec{B}$ is in the direction of $\vec{C}$, then $\vec{B} \times \vec{A}$ is in the direction of $-\vec{C}$ **so** $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

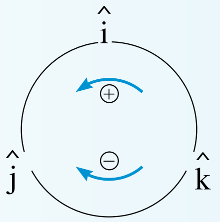
2- By applying the right hand rule on the right set of orthogonal unit vectors **then**

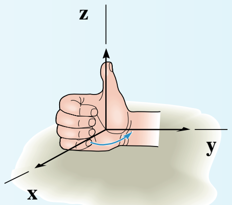
$\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$

$\hat{i} \times \hat{k} = -\hat{j}$, $\hat{k} \times \hat{j} = -\hat{i}$

$\hat{j} \times \hat{i} = -\hat{k}$





3- for any vector $\vec{A}$, $\vec{A} \times \vec{A} = \vec{O}$ Where $\vec{O}$ is the zero vector.

Example

9 $\vec{A}$ and $\vec{B}$ Are two vectors in a plane and the measure of angle between them is 70°.

If $\| \vec{A} \| = 15$, $\| \vec{B} \| = 17.5$, find the norm of $\vec{A} \times \vec{B}$

Solution

$\because \vec{A} \times \vec{B} = (\| \vec{A} \| \| \vec{B} \| \sin \theta) \vec{C}$

$\therefore \| \vec{A} \times \vec{B} \| = \| \vec{A} \| \| \vec{B} \| \sin \theta = 15 \times 17.5 \times \sin 70 = 246.67$

Try to solve

9. If $\overrightarrow{A} \times \overrightarrow{B} = -65\ \overrightarrow{C}$ and $\|\overrightarrow{A}\| = 5$, $\|\overrightarrow{B}\| = 26$, find the measure of the angle between $\overrightarrow{A}$ and $\overrightarrow{B}$

The cross production in Cartisian coordinates

If $\overrightarrow{A} = (A_x, A_y, A_Z)$, $\overrightarrow{B} = (B_x, B_y, B_Z)$ two vectors, **then**

$$\overrightarrow{A} \times \overrightarrow{B} = (A_x\ \hat{i} + A_y\ \hat{j} + A_Z\ \hat{k}) \times (B_x\ \hat{i} + B_y\ \hat{j} + B_Z\ \hat{k})$$

$$= A_x B_x\ \hat{i} \times \hat{i} + A_x B_y\ \hat{i} \times \hat{j} + A_x B_Z\ \hat{i} \times \hat{k}$$

$$+ A_y B_x\ \hat{j} \times \hat{i} + A_y B_y\ \hat{j} \times \hat{j} + A_y B_Z\ \hat{j} \times \hat{k}$$

$$+ A_Z B_x\ \hat{k} \times \hat{i} + A_Z B_y\ \hat{k} \times \hat{j} + A_Z B_Z\ \hat{k} \times \hat{k}$$

where $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \overrightarrow{O}$

$\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$, then

$$\overrightarrow{A} \times \overrightarrow{B} = 0 + A_x B_y\ \hat{k} + A_x B_Z (-\ \hat{j})$$

$$+ A_y B_x (-\ \hat{k}) + 0 + A_y B_Z (\ \hat{i}\ )$$

$$+ A_Z B_x (\ \hat{j}\ ) + A_Z B_y (-\ \hat{i}) + 0$$

$$= (A_y B_Z - A_Z B_y)\ \hat{i} + (A_Z B_x - A_x B_Z)\ \hat{j} + (A_x B_y - A_y B_x)\ \hat{k}$$

The last form can be written in a form of determinant of order 3×3 as follows:

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Spiecial case

If $\overrightarrow{A} = (A_x, A_y)$, $\overrightarrow{B} = (B_x, B_y)$ in the xy-plane, **then**

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & 0 \\ B_x & B_y & 0 \end{vmatrix} = (A_x B_y - A_y B_x)\ \hat{k}$$

Example

10 If $\overrightarrow{A} = (-2, 3, 1)$, $\overrightarrow{B} = (1, 2, 4)$ find $\overrightarrow{A} \times \overrightarrow{B}$, then deduce the unit vector perpendicular to the plane which contains the two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$

Solution

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= (3 \times 4 - 2 \times 1)\ \hat{i} - (-2 \times 4 - 1 \times 1)\ \hat{j} + (-2 \times 2 - 3 \times 1)\ \hat{k}$$

$$= 10\ \hat{i} + 9\ \hat{j} - 7\ \hat{k}$$

The perpendicular unit vector on the plane of $\overrightarrow{A}$, $\overrightarrow{B}$ $= \dfrac{\overrightarrow{A} \times \overrightarrow{B}}{\| \overrightarrow{A} \times \overrightarrow{B} \|}$

$$\therefore\ \overrightarrow{C} = \frac{10\ \hat{i} + 9\ \hat{j} - 7\ \hat{k}}{\sqrt{10^2 + 9^2 + (-7)^2}} = \frac{10}{\sqrt{230}}\ \hat{i} + \frac{9}{\sqrt{230}}\ \hat{j} - \frac{7}{\sqrt{230}}\ \hat{k}$$

Try to solve

10 If $\| \overrightarrow{A} \| = 6$ and the direction cosines of the vector $\overrightarrow{A}$ are $\frac{2}{3}$, $\frac{-2}{3}$, $\frac{1}{3}$ respectively and $\overrightarrow{B} = (-2, 3, 5)$, find $\overrightarrow{A} \times \overrightarrow{B}$

Properties of the vector product of two vectors

If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two vectors, θ is the measure of the angle between them, then:

1- $\overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{B} \times \overrightarrow{A}$ **(the cross product is not commutative)**

2- $\overrightarrow{A} \times \overrightarrow{A} = \overrightarrow{B} \times \overrightarrow{B} = \overrightarrow{O}$

3- If $\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{O}$, then either $\overrightarrow{A} \,//\, \overrightarrow{B}$ or one of them or both is equal to $\overrightarrow{O}$

4- $\overrightarrow{A} \times (\overrightarrow{B} + \overrightarrow{C}) = (\overrightarrow{A} \times \overrightarrow{B}) + (\overrightarrow{A} \times \overrightarrow{C})$ **distribution property**

5- $(k\overrightarrow{A}) \times \overrightarrow{B} = \overrightarrow{A} \times (k\overrightarrow{B}) = k(\overrightarrow{A} \times \overrightarrow{B})$ **where k is a real number**

Parallelism of two vectors

We saw in the properties of the cross multiplication that $\overrightarrow{A}$ and $\overrightarrow{B}$ are parallel if and only if:

$\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{O}$

So $(A_y B_z - A_z B_y)\ \hat{i} + (A_z B_x - A_x B_z)\ \hat{j} + (A_x B_y - A_y B_x)\ \hat{k} = \overleftarrow{O}$

So $A_y B_z = A_z B_y$, $A_x B_z = A_z B_x$, $A_x B_y = A_y B_x$

So $\frac{A_y}{B_y} = \frac{A_z}{B_z}$, $\frac{A_x}{B_x} = \frac{A_z}{B_z}$, $\frac{A_x}{B_x} = \frac{A_y}{B_y}$

So $\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$

let any of the rations = K, then

$A_x = k\,B_x, A_y = k\,B_y , A_z = k\,B_z$

$\because \overrightarrow{A} = A_x\ \hat{i} + A_y\ \hat{j} + A_z\ \hat{k}$

$\therefore \overrightarrow{A} = k\ \overrightarrow{B}$

When $k > 0$, the two vectors are parallel and in the same direction and when $k < 0$, the two vectors are parallel and in opposite directions.

Example

(11) If the vector $\overrightarrow{A} = 2\ \hat{i} - 3\ \hat{j} + m\ \hat{k}$ is parallel to vector $\overrightarrow{B} = \hat{i} + k\ \hat{j} + 8\ \hat{k}$, find value of m, k

Solution

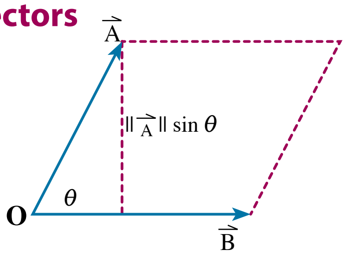
$\because \overrightarrow{A} \,//\, \overrightarrow{B}$ $\quad\quad \therefore \frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$

$\therefore \frac{2}{1} = \frac{-3}{k} = \frac{m}{8}$ $\quad\quad \therefore k = \frac{1 \times -3}{2} = \frac{-3}{2}$, $m = \frac{2 \times 8}{1} = 16$

Try to solve

(11) If $\overrightarrow{A} = (2, -3)$, $\overrightarrow{B} \,//\, \overrightarrow{A}$ and if $\| \overrightarrow{B} \| = 3\sqrt{13}$, find $\overrightarrow{B}$.

The Geometrical meaning of the vector product of two vectors



We know that $\| \overrightarrow{A} \times \overrightarrow{B} \| = \| \overrightarrow{A} \| \| \overrightarrow{B} \| \sin\theta$

$= \| \overrightarrow{B} \| \times L$

where $L = \| \overrightarrow{A} \| \sin\theta$

= The area of the parallelogram in which $\overrightarrow{B}$ and $\overrightarrow{A}$ are two adjacent sides

= double the area of triangle in which $\overrightarrow{B}$ and $\overrightarrow{A}$ are two adjacent sides

Example

(12) If $\overrightarrow{A} = (-3, 1, 2)$, $\overrightarrow{B} = (3, 4, -1)$, find the area of the parallelogram in which $\overrightarrow{A}$ and $\overrightarrow{B}$ are two adjacent sides.

Solution

$$\overrightarrow{A} \times \overrightarrow{B} = (-3, 1, 2) \times (3, 4, -1)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & 2 \\ 3 & 4 & -1 \end{vmatrix} = (-9)\,\hat{i} + 3\,\hat{j} - 15\,\hat{k}$$

$$\|\overrightarrow{A} \times \overrightarrow{B}\| = \sqrt{(-9)^2 + (3)^2 + (-15)^2} = 3\sqrt{35}$$

$\therefore$ the area of the parallelogram $= 3\sqrt{35}$ unit area.

Try to solve

(12) If $\overrightarrow{A} = (1, 2, -4)$, $\overrightarrow{B} = (0, 5, -1)$, find the area of the triangle in which $\overrightarrow{A}$, $\overrightarrow{B}$ are two sides.

Learn

The scalar triple product

If $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$ are vectors, then the expression $\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C}$ is known as the scalar triple product which has a lot of applications in the statices filed)

(notice that the expression has no brackets where doing the scalar product first is meaningless)

let $\overrightarrow{A} = (A_x, A_y, A_z)$, $\overrightarrow{B} = (B_x, B_y, B_z)$, $\overrightarrow{C} = (C_x, C_y, C_z)$

then $\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C}$

$$= (A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}) \cdot [(B_y C_z - B_z C_y)\,\hat{i} - (B_x C_z - B_z C_x)\,\hat{j}$$

$$+ (B_x C_y - B_y C_x)\,\hat{k}\,]$$

$$A_x (B_y C_z - B_z C_y) - A_y(B_x C_z - B_z C_x) + A_z(B_x C_y - B_y C_x)$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The properties of the scalat triple product

1- The value of scalar triple product does not change if the vectors are permuted in such a way that they are still read i the same cyclic order.

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

notice the cyclic order of $\vec{A}$, $\vec{B}$, $\vec{C}$

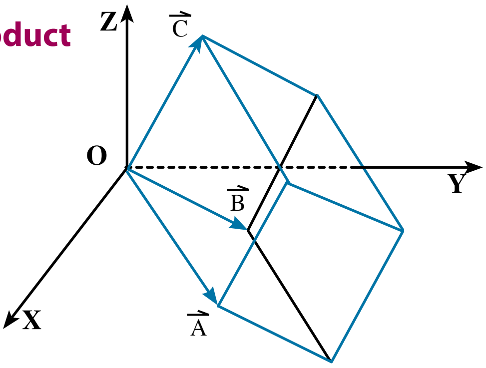
2- If the vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ in the same plane, then the scalar triple product vanishes

So $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$

The geometrical meaning of the scalar triple product

If $\vec{A}$, $\vec{B}$, $\vec{C}$ are 3 vectors forming 3-non parallel sides of a parallelepiped, then the volume of the parallel piped = the absolute value of the scalar triple product.

Thus the volume of the parallelepiped = $|\vec{A}\cdot\vec{B}\times\vec{C}|$



Example

(13) Find the volume of the parallelepiped in which three adjacent sides are represented by the vectors $\vec{A}$ = (2, 1, 3), $\vec{B}$ = (-1 , 3 , 2), $\vec{C}$ = (1 , 1 , -2)

Solution

The volume of the parallelepiped = $|\vec{A}\cdot\vec{B}\times\vec{C}|$ **(1)**

$$\therefore\ \vec{A}\cdot\vec{B}\times\vec{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & 2 \\ 1 & 1 & -2 \end{vmatrix} = -28$$

then the volume of the parallelepiped= |-28| = 28 unit volume

Try to solve

(13) Find the volume of the parallelepiped in which three- non parallel edges are represented by the vectors $\vec{A}$ = (3 , -4 , 1), $\vec{B}$ = (0 , 2 , -3) , $\vec{C}$ = (3 , 2, 2)

Exercises 1 - 3

Complete the following: If $\hat{i}$, $\hat{j}$, $\hat{k}$ **form a right hand system of unit vectors** :

1. $\hat{i} \cdot \hat{j}$ = ...

2. $\hat{j} \times \hat{k}$ = ...

3. If $\overrightarrow{A}$ = (2 , -1) , $\overrightarrow{B}$ = (3 , -4), then the component of $\overrightarrow{A}$ in the direction of $\overrightarrow{B}$ equals ....

4. If $\overrightarrow{A}$, $\overrightarrow{B}$ are non zero vectors and $\overrightarrow{A} \cdot \overrightarrow{B} = 0$, then $\overrightarrow{A}$, $\overrightarrow{B}$ are ....

5. If $\overrightarrow{A}$, $\overrightarrow{B}$ are non zero vectors and $\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{O}$, then $\overrightarrow{A}$, $\overrightarrow{B}$ are ....

6. The measure of the angle between the two vectors $3\hat{i} - \hat{j}$, $-4\hat{i} + 6\hat{j}$ equals ....

7. The work done by the force $\overrightarrow{F} = 3\hat{i} + 7\hat{k}$ to move a body from point A(1 , 1, 2) to point B(7 , 3, 5) equals ....

Choose the correct answer from the following:

8. $\hat{i} \times \hat{j}$ = ...

   a) $\overrightarrow{O}$ b) 0 c) 1 d) $\hat{k}$

9. If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two perpendicular unit vectors, then $(\overrightarrow{A} - 2\overrightarrow{B}) \cdot (3\overrightarrow{A} + 5\overrightarrow{B})$ =

   a) -8 b) -7 c) 24 d) 0

10. If $\overrightarrow{A}$ and $\overrightarrow{B}$ are unit vectors, then $\overrightarrow{A} \cdot \overrightarrow{B} \in$ ....

    a)]0, 1[b)]-1, 1[c) [-1, 1] d) R^+

11. The measure of the angle between the two vectors (2 , -2, 2) , (1, 1, 4) is ....

    a) 57.02° b) 35.26° c) 134.37° d) 0°

12. If the vectors (2 ,k, -3) , (4 , 6, -6) are parallel, then k = ...

    a) 6 b) 3 c) -3 d) 1

Answer the following:

13. Find $\overrightarrow{A} \cdot \overrightarrow{B}$ in each of the following:

    a) $\overrightarrow{A}$ = (5 , 1 , -2) , $\overrightarrow{B}$ = (4 , -4, 3)

    b) $\overrightarrow{A} = -3\hat{i} - 2\hat{j} - \hat{k}$, $\overrightarrow{B} = 6\hat{i} + 4\hat{j} + 2\hat{k}$

    c) $\overrightarrow{A} = \hat{i}$, $\overrightarrow{B} = 2\hat{j} - \hat{k}$

14 Find the measure of the angle between the two vectors in each of the following:

a) (5 , 1, -2) , (1 , 1, -1)

b) (7 , 2 , -10) , (2 , 6 , 4)

c) (2 , 1 , 4) , (1, -2, 0)

15 Find $\overrightarrow{A} \times \overrightarrow{B}$ in each of the following:

a) $\overrightarrow{A} = (-2, 3, 1)$, $\overrightarrow{B} = (1, 3, -4)$

b) $\overrightarrow{A} = -\hat{i} - 2\hat{j}$, $\overrightarrow{B} = 3\hat{j} - 5\hat{k}$

c) $\|\overrightarrow{A}\| = 6$, $\|\overrightarrow{B}\| = 8$ the angle between them is $60°$

16 ABCD is a square with side length 12cm. $\hat{e}$ is the unit vector perpendicular to its plane, find:

a) $\overrightarrow{AB} \cdot \overrightarrow{AC}$

b) $\overrightarrow{AB} \times \overrightarrow{CA}$

c) $\overrightarrow{BC} \cdot \overrightarrow{AD}$

d) $\overrightarrow{BD} \times \overrightarrow{AC}$

e) $\overrightarrow{AB} \cdot \overrightarrow{BC}$

f) $\overrightarrow{AB} \times \overrightarrow{BC}$

17 Find the unit vector perpendicular to the plane which contains the two vectors.

$\overrightarrow{A} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\overrightarrow{B} = -\hat{i} + 2\hat{j} - 3\hat{k}$

18 Calculate the area of the triangle DEF in each of the following:

a) D(5 , 1 , -2) , E (4 , -4 , 3) , F (2 , 4 , 0)

b) D (4 , 0 , 2) , E (2 , 1 , 5) , F (-1 , 0 , -1)

19 Calculate the area of the parallelogram LMNE in each of the following :

a) L (1 , 1) , M (2 , 3) , N(5 , 4)

b) L (2 , 1 , 3) , M (1 , 4 , 5) , N(2 , 5 , 3)

20 Find volume of the parallelepiped in which $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$ are three adjacent edges:

$\overrightarrow{A} = (1, 1, 3)$, $\overrightarrow{B} = (2, 1, 4)$, $\overrightarrow{C} = (5, 1, -2)$

21 In each of the following, show whether the two given vectors are parallel or perpendicular or otherwise:

a) $\overrightarrow{A} = (0, 2, 2)$, $\overrightarrow{B} = (3, 0, -4)$

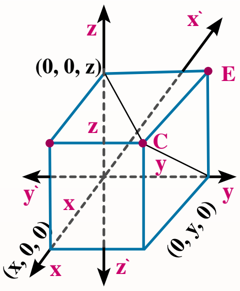
b) $\overrightarrow{E} = 10\hat{i} + 40\hat{j}$, $\overrightarrow{F} = -3\hat{j} + 8\hat{k}$

c) $\overrightarrow{A} = -2\hat{i} + \hat{j} - 2\hat{k}$, $\overrightarrow{B} = 8\hat{i} - 4\hat{j} + 8\hat{k}$

Unit Summry

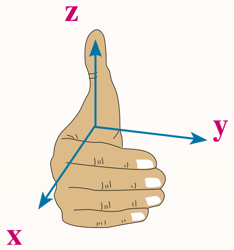
The 3D - orthogonal coordinate system

Identifying the coordinates of point A in space by knowing its projection on each axis of the coordinate axes .



Right hand rule

Where the curved fingers refer from the +ve direction of x-axis towards the +ve direction of y axis and the thump shows the +ve direction of z-axis



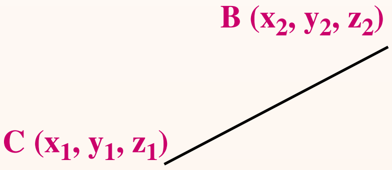
Cartesian planes

- The cartisian palne xy its equation is $Z = 0$
- The cartisian palne xz its equation is $y = 0$
- The cartisian palne yz its equation is $x = 0$

The distance between two points

If $A(x_1, y_1, z_1)$, $b\ (x_2, y_2, z_2)$ are two points in space, then the length of the line segment $\overline{AB}$ is given by

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$



The coordinates of the midpoint of a line segment

If $A(x_1, y_1, z_1)$ and $B\ (x_2, y_2, z_2)$ are two points in space, then the coordinates of the midpoint of $\overline{AB}$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

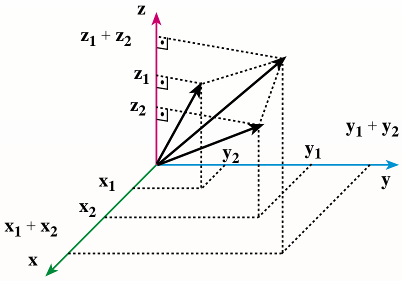
The equation of the sphere in space

the equation of sphere whose center is (l, k, n) and its radius length is r is

$(x - L)^2 + (y - k)^2 + (z - n)^2 = r^2$

- The equation of the sphere whose center is the origin and radius length r is $x^2 + y^2 + z^2 = r^2$
- The equation of the sphere : $x^2 + y^2 + y^2 + 2\ \ell x + 2\ ky + 2\ nz + d = 0$ where its center is $(-\ell , - k , - n)$ and length of its radius length is $(r) = \sqrt{\ell^2 + k^2 + n^2 - d}$ where $\ell^2 + k^2 + n^2 > d$

(The position vector in space



If $A(A_x, A_y, A_z)$ is a point in space, then the position vector of point A with respect to the origin $\overrightarrow{A} = (A_x, A_y, A_z)$

- A_x is called the component of the vector $\overrightarrow{A}$ in the direction of x-axis
- A_y is called the component of the vector $\overrightarrow{A}$ in the direction of y-axis
- A_z is called the component of the vector $\overrightarrow{A}$ in the direction of z-axis

The norm of a vector

If $\overrightarrow{A} = (A_x, A_y, A_z)$ then $\|\overrightarrow{A}\| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$

Adding and subtracting vectors in space

If $\overrightarrow{A} = (A_x, A_y, A_Z)$

$\overrightarrow{B} = (B_x, B_y, B_Z)$, then:

1- $\overrightarrow{A} + \overrightarrow{B} = (A_x + B_x, A_y + B_y, A_Z + B_Z)$

2- $\overrightarrow{A} - \overrightarrow{B} = (A_x - B_x, A_y - B_y, A_Z - B_Z)$

Properties of addition

1- $\overrightarrow{A} + \overrightarrow{B} \in R^3$ — **Closure property**

2- $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{B} + \overrightarrow{A}$ — **Commutative property**

3- $(\overrightarrow{A} + \overrightarrow{B}) + \overrightarrow{C} = \overrightarrow{A} + (\overrightarrow{B} + \overrightarrow{C})$ — **Associative property**

4- $\overrightarrow{A} + \overrightarrow{O} = \overrightarrow{O} + \overrightarrow{A} = \overrightarrow{A}$ — **The identy element of addition**

5- $\overrightarrow{A} + (-\overrightarrow{A}) = (-\overrightarrow{A}) + \overrightarrow{A} = \overrightarrow{O}$ — **The additive inverse**

Multiplying a vector by a real number

If $\overrightarrow{A} = (A_x, A_y, A_Z)$, $k \in R$ then $k\overrightarrow{A} = (kA_x, kA_y, kA_Z)$

Equality of vectors in space

If $(A_x, A_y, A_Z) = (B_x, B_y, B_Z)$, then: $A_x = B_x$, $A_y = B_y$, $A_Z = B_Z$

The unit vector

The unit vector is a vector whose norm is one unit length

The fundamental unit vectors

$\hat{i} = (1, 0, 0)$ The unit vector in the +ve direction of x-axis

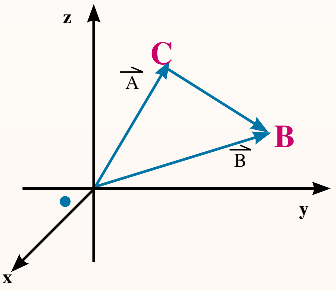
$\hat{j} = (0, 1, 0)$ The unit vector in the +ve direction of y-axis

$\hat{j} = (0, 0, 1)$ The unit vector in the +ve direction of z-axis

Expressing a vector in terms of the fundamental unit vectors

If $\vec{A} = (A_x, A_y, A_z)$, then we can write the vector $\vec{A}$ in the form of $\vec{A} = A_x\ \hat{i} + A_y\ \hat{j} + A_z\ \hat{k}$

Expressing the directed line segment in space in terms of the coordinates of its terminals



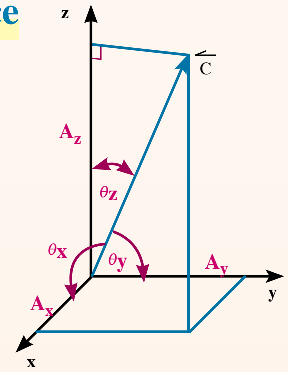
If A and B are two points in space their position vectors are $\vec{A}$ and $\vec{B}$ respectively, then $\vec{AB} = \vec{B} - \vec{A}$

The unit vector in the direction of a given vector

If $\vec{A} = (A_x, A_y, A_z)$, then the vector $\vec{U_A}$ is called the unit vector in the direction of $\vec{A}$

$$\overleftarrow{U_A} = \frac{\vec{A}}{\| \vec{A} \|}$$

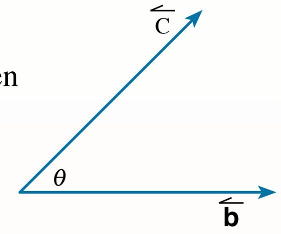
Direction angles and direction cosine of a vector in space



If $(\theta_x, \theta_y, \theta_z)$ are the measures of angles among the vector $\vec{A} = (A_x, A_y, A_z)$ and the +ve directions of x , y , z axes respectively, then:

- $A_x = \| \vec{A} \| \cos\theta_x$, $A_y = \| \vec{A} \| \cos\theta_y$, $A_z = \| \vec{A} \| \cos\theta_z$
- $(\theta_x, \theta_y, \theta_z)$ is called direction angles of vectors $\vec{A}$
- $\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$ is called direction cosines of the vector $\vec{A}$
- $\cos\theta_x\ \hat{i} + \cos\theta_y\ \hat{j} + \cos\theta_z\ \hat{k}$ represents the unit vector in the direction of $\vec{A}$
- $\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$

The scaler product of two vectors



If $\vec{A}$ and $\vec{B}$ are two vectors in R^3 and the measure of the angle between them is θ where $0 \leqslant \theta \leqslant 180°$, then $\vec{A} \cdot \vec{B} = \| \vec{A} \| \| \vec{B} \| \cos \theta$

The properties of the scaler product of two vectors

1- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ — **Commutative properties**

2- $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$ — **Distributive properties**

3- If k is a real number, then $(k\vec{A}) \cdot \vec{B} = \vec{A} \cdot (k\vec{B}) = k(\vec{A} \cdot \vec{B})$

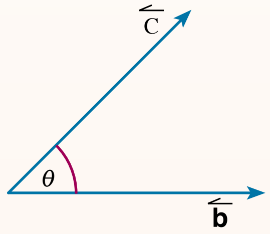
4- $\vec{A} \cdot \vec{A} = \| \vec{A} \|^2$

5- If $\vec{A} \cdot \vec{B} = 0$ then $\vec{A} \perp \vec{B}$

The scaler product of two vectors in an orthogonal coordinate system

If $\vec{A} = (A_x , A_y, A_Z)$, $\vec{B} = (B_x , B_y, B_Z)$ then $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_Z B_Z$

The angle between two vectors

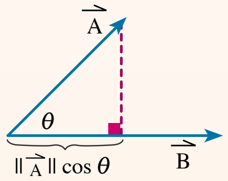


$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{\| \vec{A} \| \| \vec{B} \|}$$

- If $\cos \theta = 1$, then $\vec{A} \,//\, \vec{B}$ and on the same direction
- If $\cos \theta = -1$, then $\vec{A} \,//\, \vec{B}$ and on the opposite direction
- If $\cos \theta = 0$, then $\vec{A} \perp \vec{B}$

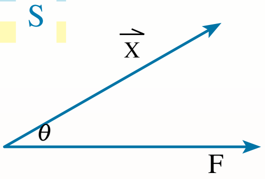
The component of a vector in the direction of another vector

The component of vector $\vec{A}$ in the direction of $\vec{B}$ is denoted by A_b



$$A_B = \| \vec{A} \| \cos \theta = \frac{\vec{A} \cdot \vec{B}}{\| \vec{B} \|}$$

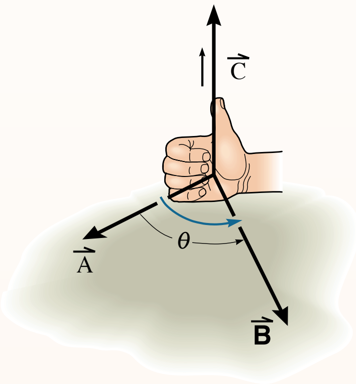
The work done by the force $\overrightarrow{F}$ to make a displacement $\overrightarrow{S}$



The work = $\overrightarrow{F} \cdot \overrightarrow{S} = \| \overrightarrow{F} \| \| \overrightarrow{S} \| \cos \theta$

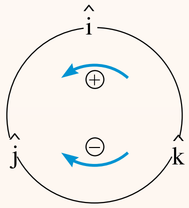
- If the force $\overrightarrow{F}$ is in the direction of the displacement ($\theta = 0°$),
  then $W = \| \overrightarrow{F} \| \| \overrightarrow{S} \|$
- If the force $\overrightarrow{F}$ is in the opposite direction of the displacement ($\theta = 180°$),
  then $W = -\| \overrightarrow{F} \| \| \overrightarrow{S} \|$
- If the force $\overrightarrow{F}$ is perpendicular to the direction of the displacement ($\theta = 90°$),
  then $W = 0$

The vector product of two vectors



If $\overrightarrow{A}$ and $\overrightarrow{B}$ are vectors in R^3 and the measure of the smallest angle between them is θ, then $\overrightarrow{A} \times \overrightarrow{B} = (\| \overrightarrow{A} \| \| \overrightarrow{B} \| \sin \theta) \overrightarrow{C}$ where $\overrightarrow{C}$ is perpendicular unit vector to the plane of $\overrightarrow{A}$ and $\overrightarrow{B}$. The direction of $\overrightarrow{C}$ is identified (up or down) According to the right hand rule where the curved fingers of the right hand to the direction of rotation from $\overrightarrow{A}$ to $\overrightarrow{B}$ and the thump shows the direction of $\overrightarrow{C}$

The properties of the vector product of two vectors



1- $\overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{B} \times \overrightarrow{A}$

2- $\overrightarrow{A} \times \overrightarrow{A} = 0$

3- $\overrightarrow{A} \times (\overrightarrow{B} + \overrightarrow{C}) = \overrightarrow{A} \times \overrightarrow{B} + \overrightarrow{A} \times \overrightarrow{C}$ distributive property

4- If $\overrightarrow{A} \times \overrightarrow{B} = 0$, then $\overrightarrow{A} // \overrightarrow{B}$ or one of the two vectors or both of them equals $\overrightarrow{O}$

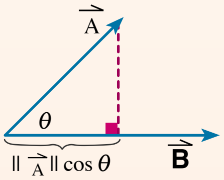
5- $\hat{i} \times \hat{j} = \hat{k}$, $\quad \hat{j} \times \hat{k} = \hat{i}$, $\quad \hat{k} \times \hat{i} = \hat{j}$

$\hat{i} \times \hat{k} = -\hat{j}$, $\quad \hat{k} \times \hat{j} = -\hat{i}$, $\quad \hat{j} \times \hat{i} = -\hat{k}$

The vector product of two vectors in a perpendicular coordinate system

If $\overrightarrow{A} = (A_x, A_y, A_Z)$, $\overrightarrow{B} = (B_x, B_y, B_Z)$ then

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



Special case : The vector product in the xy-plane

If $\overrightarrow{A} = (A_x , A_y)$, $\overrightarrow{B} = (B_x , B_y)$, then:

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{k} = (A_x B_y - A_y B_x) \hat{k}$$

The perpendicular unit vectors on the plane of the vectors $\overrightarrow{A}$, $\overrightarrow{B}$

$$\overrightarrow{C} = \frac{\overrightarrow{A} \times \overrightarrow{B}}{\| \overrightarrow{A} \times \overrightarrow{B} \|}$$

Parallelism of two vectors

The two vectors $\overrightarrow{A} =(A_x, A_y, A_z)$, $\overrightarrow{B} = (B_x , B_y, B_z)$ are parallel if one of the following conditions occures:

1- $\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{O}$

2- $\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$

3- $\overrightarrow{A} = k \overrightarrow{B}$

If $k > 0$, then the two vectors $\overrightarrow{A}$, $\overrightarrow{B}$ are parallel and in the same directions.

If k 0 > ,then the vectors $\overrightarrow{A}$, $\overrightarrow{B}$ are parallel and in the opposite directions.

The geometrical meaning of vector product

$\| \overrightarrow{A} \times \overrightarrow{B} \|$ = the area of the parallelogram where $\overrightarrow{A}$ and $\overrightarrow{B}$ are two adjacent sides.

= double the area of triangle where $\overrightarrow{A}$ and $\overrightarrow{B}$ two sides.

The scalar triple product

$$\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The geometrical meaning of the scalar triple product

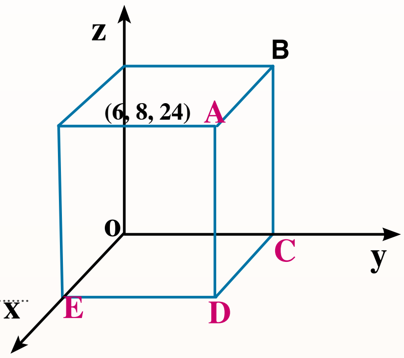
the volume of parallelepiped where $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$ are three vectors represent the non parallel edges equals the absolute value of $\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C}$

General exercises

Complete the following:

1. The point (2 , 0 , -3) lies in the coordinates plane .... whose equation is ....

2. The opposite figure represents the a cuboid:
   A (6 , 8 , 24) , then



   a. The coordinates of D is ....
   b. The coordinates of C is ....
   c. The direction angles of the vector $\overrightarrow{OD}$ is ....

3. If A(-1 , 2 , 3) , B(4 , -1 , 5), then $\overrightarrow{AB}$ = ...

4. If point (-2 , 4 , m) lies on the sphere $(x + 2)^2 + (y - 1)^2 + (z - 3)^2 = 25$, then m = ...

5. If $\overrightarrow{A}$ = (k, 3, -4) , $\overrightarrow{B}$ = (-2 , 9 , m) and $\overrightarrow{A}$ // $\overrightarrow{B}$, then k = ... , m = ...

6. If $\overrightarrow{A} \cdot \overrightarrow{B} = \| \overrightarrow{A} \times \overrightarrow{B} \|$, then the measure of the angles between the two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ equals ....

Choose the correct answer from the following:

7. The straight lines $\overleftrightarrow{xx'}$, $\overleftrightarrow{zz'}$ form the coordinate plane whose equation is ....
   a. x = 0 b. y = 0 c. z = 0 d. y = 2

8. The equation of the sphere whose center is the origin and passes through (3 , -1, 2) is ......
   a. $x^2 + y^2 + z^2 = 4$
   b. $(x - 3)^2 + (y + 1)^2 + (z - 2)^2 = 14$
   c. $(x - 3)^2 + (y + 1)^2 + (z - 2)^2 = \sqrt{14}$
   d. $x^2 + y^2 + z^2 = 14$

9. The coordinates of the midpoint of $\overline{DE}$ where D(2 , 3 , 3) , E (6 , -1 , -4) is
   a. $(4 , 2 , 3\frac{1}{2})$ b. $(2 , 1 , \frac{1}{2})$ c. $(4 , 1 , -\frac{1}{2})$ d. $(4 , 1 , \frac{1}{2})$

10. If $\overrightarrow{A}$ // $\overrightarrow{B}$, then $\| \overrightarrow{A} \times \overrightarrow{B} \|$= ....
    a. 0 b. 1 c. $\| \overrightarrow{A} \|$ d. $\| \overrightarrow{B} \|$

11. The vector which represents the unit vector of the following vectors is:
    a. (-3 , 2 , 2) b. $(\frac{1}{2} , \frac{1}{3} , \frac{1}{4})$ c. $(\frac{3}{5} , \frac{-4}{5} , 0)$ d. $(\frac{-1}{12} , \frac{-3}{12} , \frac{2}{12})$

12 If ℓ, m, n are the direction cosine angles of vectors $\overrightarrow{A}$ then:

a $\ell + m + n = 1$ b $\ell = m = n$ c $\ell^2 + m^2 + n^2 = 1$ d $\ell + m + n = \|\overrightarrow{A}\|$

13 If $\overrightarrow{A} \times \overrightarrow{C} = \overrightarrow{O}$, $\overrightarrow{A} \cdot \overrightarrow{B} = 0$, then $\overrightarrow{B} \cdot \overrightarrow{C} =$ ....

a 0 b 1 c $\overrightarrow{A}$ d $\|\overrightarrow{B}\|$

14 xy-plane and yz-planes intersect at ....

a origin point b x-axis c y-axis d z-axis

Answer the following:

15 Prove that the triangle whose vertices are (7 , 1 , 3), (5 , 3 , 4), (3 , 5 , 3) is an isoscles triangle .

16 Find the center and the radius length of the sphere $x^2 + y^2 + z^2 = 6z$

17 If A (-2 , 3, 5), B (1 , 4 , -2) ,find $\overrightarrow{AB}$

18 If $\overrightarrow{C} = (1 , -2 , 2)$, find the unit vector in the direction of $\overrightarrow{C}$

19 If the vector $\overrightarrow{A}$ makes with the (+ve) direction of the coordinate axes x, y, z angles of measure 60° , 80° , θ° where θ is an Acute angle

a Find value of θ

b Write the Algebraic from of $\overrightarrow{A}$ if $\|\overrightarrow{A}\| = 13$

20 If $\overrightarrow{A} = (1 , 6 , 2)$, $\overrightarrow{B} = (k, 3 , m)$, $\overrightarrow{C} = (k, m , k + m)$ and $\overrightarrow{A} \parallel \overrightarrow{B}$, find $\|\overrightarrow{C}\|$

21 If $\overrightarrow{A}$ and $\overrightarrow{B}$ are two unit vectors in R^3, at what condition does the result of the vector product $\overrightarrow{A} \times \overrightarrow{B}$ represent a unit vector in R^3. Explain your Answer.

22 ABCD is rectangle in which AB = 6cm, BC = 8cm, find:

a $\overrightarrow{AB} \cdot \overrightarrow{AC}$ b $\overrightarrow{AB} \cdot \overrightarrow{CD}$

c the componant of $\overrightarrow{CD}$ in the direction of $\overrightarrow{BC}$

23 Find the work done by the force $\overrightarrow{F} = (2 , -3, 5)$ to move A body from point (1 , -1 , 0) to point (2 , 4 , -2)

24 Find the work done by the weight of a body of magnitude 40 N in projected vertically upwards for a distance of 10 meters above the surface of the ground

25 Prove each of the following where $\overrightarrow{A}$, $\overrightarrow{B}$, $\in R^3$

a $\|\overrightarrow{A} \times \overrightarrow{B}\|^2 + |\overrightarrow{A} \cdot \overrightarrow{B}|^2 = \|\overrightarrow{A}\|^2 \|\overrightarrow{B}\|^2$

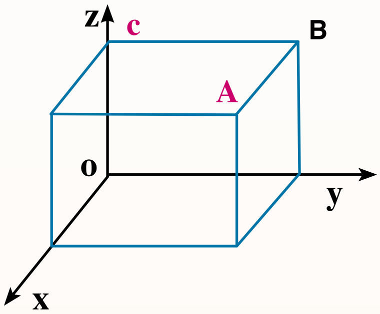
b If $\overrightarrow{A} \cdot \overrightarrow{B} = 0$, $\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{O}$ then $\overrightarrow{A} = \overrightarrow{O}$ or $\overrightarrow{B} = \overrightarrow{O}$

For more activities and exercise, visit www.sec3mathematics.com.eg

Accumulative test

Complete the following:

1. The distance between the point (-4 , -3, 2) and yz-plane is .... unit length
2. If (a - 2, b + 3 , c - 4) lies in xz-plane, then b = ...
3. If A (-1 , 0 , -2) , B (5 , -1 , 4), then the midpoint of $\overline{AB}$ is ....
4. The opposite figure is a cuboid : A (5, 8 , 4), then
   - a) The coordinates of the point B is ....
   - b) The coordinates of point C is ....



5. The equation of the sphere with center (1, -3, -1) and passes through the point (-2, -1, -1) is ....
6. If $\overrightarrow{A} = (-2, 3, 1)$, $\overrightarrow{B} = (0, 2, -2)$, $\overrightarrow{C} = (1, -3, 5)$, then $2\overrightarrow{A} - 3\overrightarrow{B} + \overrightarrow{C} =$ ....
7. If $\overrightarrow{A} = (5, -2, 3)$, then the unit vector in the direction of $\overrightarrow{A}$ is ....

Choose the correct answer:

8. If $\overrightarrow{A} = (3, -2, m)$, $\|\overrightarrow{A}\| = \sqrt{22}$, then m = ...
   - a) 21
   - b) ± 9
   - c) ± 3
   - d) 17
9. If $\overrightarrow{A} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\overrightarrow{B} = 4\hat{i} - \hat{j}$, then $\overrightarrow{A} \cdot \overrightarrow{B} =$
   - a) 5
   - b) 4
   - c) ± 3
   - d) 2
10. If $\overrightarrow{A} = 4\hat{i} - 3\hat{j} + 5\hat{k}$, then the component of $\overrightarrow{A}$ in the direction of y-axis equals
    - a) 4
    - b) 3
    - c) -3
    - d) 5
11. If $\overrightarrow{A} = (2, 3, -1)$, $\overrightarrow{B} = 3\hat{i} + 4\hat{j}$, then the component of $\overrightarrow{A}$ in the direction of $\overrightarrow{B}$ equals
    - a) 18
    - b) $\frac{18}{5}$
    - c) $-\frac{18}{5}$
    - d) $\frac{18}{25}$
12. If the direction angles of a vector are 45° , 45° , θ , then $\theta =$ ....
    - a) 45°
    - b) 90°
    - c) 0°
    - d) 60°

Answer the following:

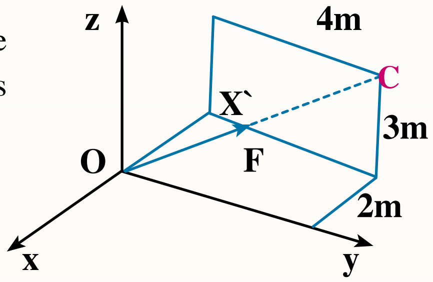
13 Prove that the triangle whose vertices are (2 , 2 , 1), (0 , 0 , 0), (2 , -4 , 4) is a right angled triangle , then find its Area.

14 Find the equation of the sphere whose center is (0 , 4, 0) and touches xz-plane

15 If $\overrightarrow{A} = (1, -3, 2)$, $\overrightarrow{B} = (0, 2, 3)$, find $\|\overrightarrow{A}\|$, $\|\overrightarrow{A} + \overrightarrow{B}\|$

16 Find the algebraic form of the vector $\overrightarrow{A}$ whose norm is $21\sqrt{3}$ and makes equal measured angles with the +ve direction of the coordinate axes.

17 In the opposite figure, find the components of the force $\overrightarrow{F}$ whose magnitude is $12\sqrt{29}$ newtens in the dirctions of the coordinate axes .



18 If $\overrightarrow{A} = \hat{i} + \hat{j} - 2\hat{k}$, $\overrightarrow{B} = 3\hat{i} - 2\hat{j} + 5\hat{k}$, find

a $\overrightarrow{A} \times \overrightarrow{B}$ b $(2\overrightarrow{A}) \times (3\overrightarrow{B})$ c $\overrightarrow{A} \times (\overrightarrow{A} - 2\overrightarrow{B})$

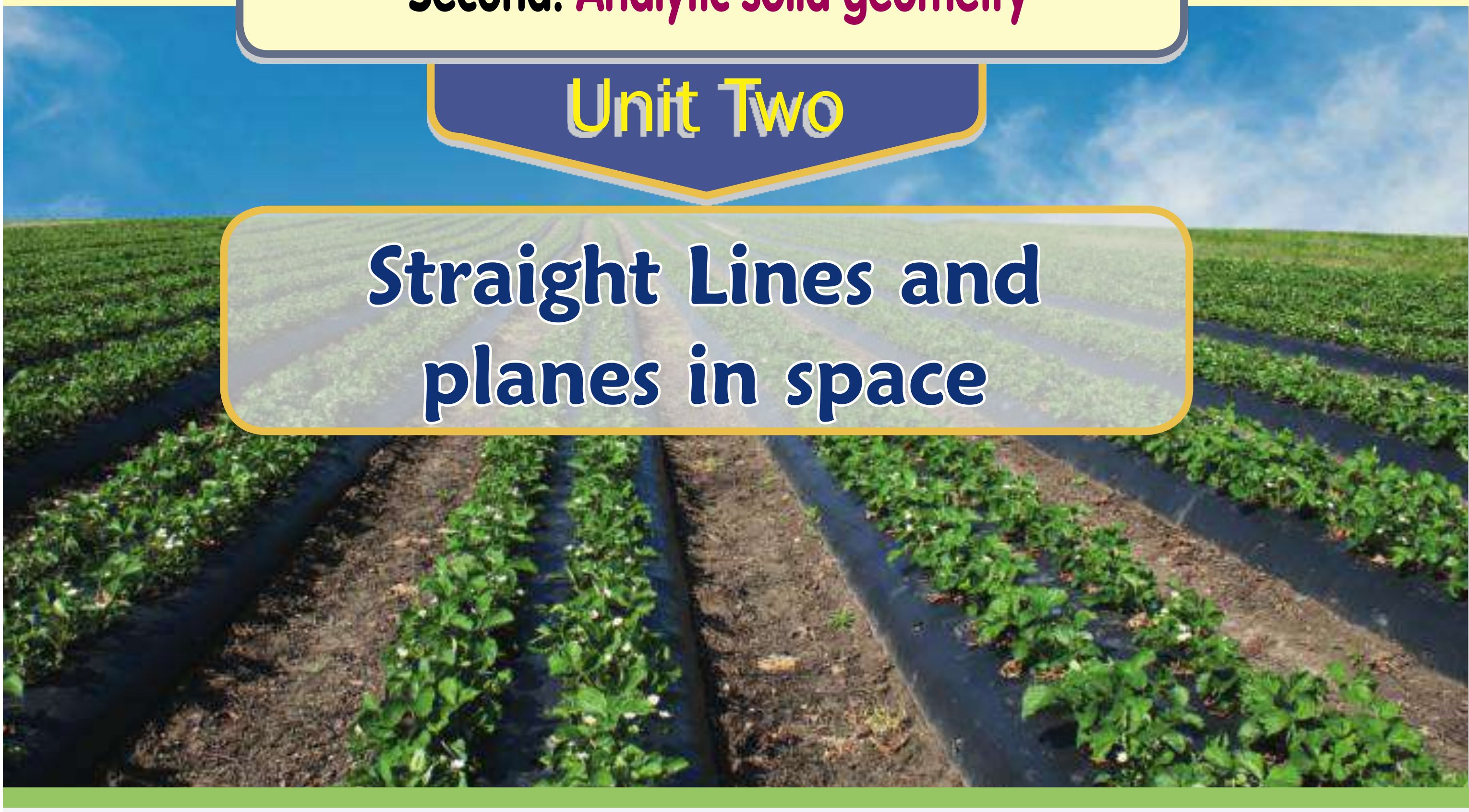
19 **Creative thinking:** If A (0 , 0, 1) , B (1 , 0 , 0), C (0 , 1 , 0), find the orthogonal unit vector to the plane A B C

20 Find the direction angles of the vector $\overrightarrow{C} = 3\hat{i} - 4\hat{j} + 5\hat{k}$.

Second: Analytic solid geometry

Unit Two

Straight Lines and planes in space



Unit introduction

In the previous unit , you learned how to identify a point in space and the position vectors and how to find their norms. These topics are fundamental in this unit since they are complementaries to what you learned in the previous unit and to what you learned last year. In this unit, you are going to learn the equation of the straight line in space and the equation of a plane in its different forms. The examples and how to solve them have been varied in order to fulfill the cognitive and skillful objectives which help the student learn the other concepts and knowledge related to the solid geometry in the next educational stages.

Unit obects

By the end of this unit and doing all activities included , the student should be able to:

- Find the direction vector of a straight line in space
- Find the parametric equations and the vector equation of a straight line in space
- Find the cartesian equation of a straight line in space
- Find the general equation of a plane in space
- Find the standard form of the equation of a plane in space
- Recognize the angle between two planes in space
- Conclude the condition of perpendicularity of two planes in space
- Conclude the condition of parallelism of two planes in space
- Find the equation of the intersection line of two planes in space
- Find the distance between a point and a straight line in space
- Find the distance between a point and a plane using the scalar product and the cartesian form
- Identify the distance between two parallel planes

Key terms

- Direction vector
- Direction angles
- Direction cosines
- Direction ratios
- Vector equation
- Parametric equations
- Cartesian equation
- General equation
- Proportional
- Parallel straight Lines
- Perpendicular straight Lines
- Intersecting straight Lines
- Skew straight Lines
- Perpendicular distance
- plane
- Standard form
- Parallel planes
- Perpendicular planes
- Intersecting planes
- Angle

Materials

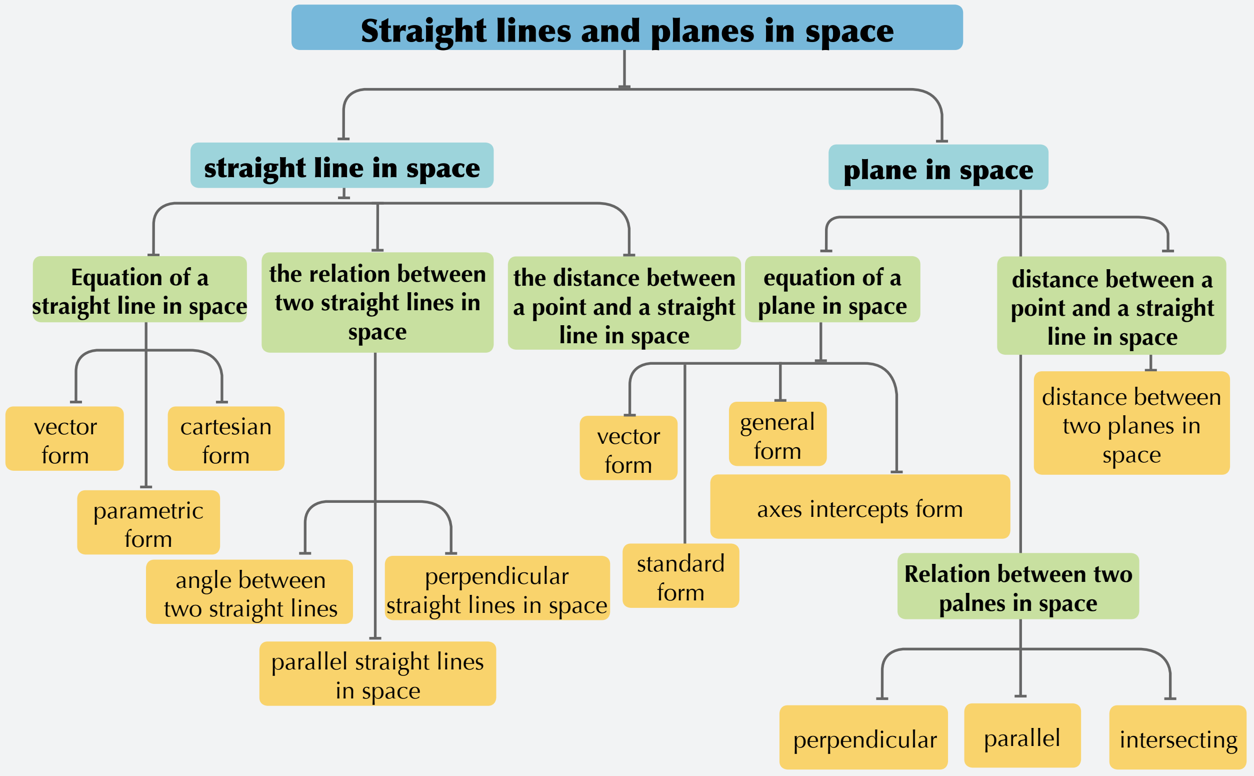
- Scientific calculator
- Computer 3D-program.

Unit Lessons

Lesson (2 - 1): The equation of a straight line in space.

Lesson (2 - 2): The equation of a plane in space.

Unit chart



Unit Two

2 - 1

Equation of a straight Line in space

You will learn

- Direction vector of a straight line
- Different forms of the equation of a straight line
- The angle between two straight lines
- The distance between a point and a straight line
- Parallel straight lines
- perpendicular straight lines

Key terms

- Direction vector
- Direction ratio
- Direction angles
- Cartesian equation
- Parametric equation

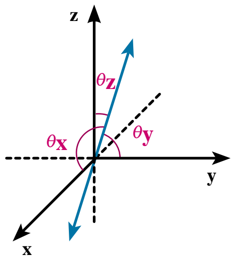
Materials

- Scientific calculator
- Computer 3-D program

You learned in the previous years the straight line in a plane and how to find the different forms of the equation of the straight line in a plane (vector -parametric-general) forms. In this lesson, we learn the straight line in space and how to get the equation of the straight line in space in different forms because it is much important in geometrical fields, architectural design and applications of space science.

Learn

Direction vector of a straight Line in space



If $\theta_x, \theta_y, \theta_z$ are the directed angles of a straight line in space, **then** $\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$ are directed cosines of these straight lines and they are usually denoted by ℓ , m , n.

$\ell = \cos\theta_x$, $m = \cos\theta_y$, $n = \cos\theta_z$

so $\ell^2 + m^2 + n^2 = 1$

so the vector $\vec{u} = \ell\,\hat{i} + m\,\hat{j} + n\,\hat{k}$ is the unit vector in the direction of the straight line.

And any vector parallel to the unit vector $\vec{u}$ is called the direction vector of the straight line and is denoted by $\vec{d}$

i.e. $\vec{d} = k(\ell\,\hat{i} + m\,\hat{j} + n\,\hat{k}) = (a, b, c)$

where a , b , c are proportional to ℓ , m , n , $k \in R^+$

a , b , c are called direction ratio (direction numbers)

for example: if $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ are the direction cosines of the straight line.

then the vector $\vec{d} = k(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$ represents a direction vector of the straight line where $k \neq 0$

put $k = 3 \longrightarrow \vec{d} = (2, 1, 2)$

put $k = -6 \longrightarrow \vec{d} = (-4, -2, -4)$

i.e the straight line has infinite number of parallel direction vectors each is parallel to the straight line.

Example

1 Find a direction vector of the straight line passing through A (-2 , 3, 1) , B (0 , 4 , -2)

Solution

A direction vector of the straight line $= \overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (0 , 4 , -2) - (-2 , 3 , 1)$

$\therefore \overrightarrow{d} = (2 , 1 , -3)$

Try to solve

1 Find a direction vector for each of the following straight lines:

a The straight line passing through the origin point and point (-1 , 2 , -2)

b The straight line passing through points C (0 , -2 , 3) and D (1 , 1 , -1)

Critical thinking:

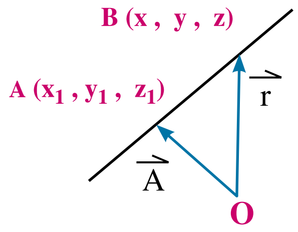
1- What can you say about the straight line with direction vector $\overrightarrow{d} = (a , b , 0)$?

2- Find a direction vector for each of the cartesian axes.

Learn

Vector form of the equation of a straight Line in space

If L is a straight line in space whose direction vector is $\overrightarrow{d} = (a , b , c)$ and passes through the point A whose position vector is $\overrightarrow{A} = (x_1, y_1, z_1)$ and if point B is any point on the straight line whose position vector is $\overrightarrow{r} = (x , y , z)$, **then**



From the figure: $\overrightarrow{r} = \overrightarrow{A} + \overrightarrow{AB}$

but $\overrightarrow{AB} \,//\, \overrightarrow{d}$ $(\overrightarrow{AB} = t\,\overrightarrow{d})$

$\therefore$ $\overrightarrow{r} = \overrightarrow{A} + t\,\overrightarrow{d}$ ⟶ vector form of the equation of straight line.

Example

2 Find the vector form of the equation of the straight line passing through point (3, -1, 0) and the vector (-2, 4 , 3) is a direction vector for it.

Solution

(3 , -1 , 0) **represents a point on the straight line** $\therefore \overrightarrow{A} = (3 , -1 , 0)$

(-2 , 4 , 3) **represents the direction vector of the straight line** $\therefore \overrightarrow{d} = (-2 , 4 , 3)$

the equation of the straight line is $\overrightarrow{r} = \overrightarrow{A} + t\,\overrightarrow{d}$

$\therefore \overrightarrow{r} = (3 , -1 , 0) + t\,(-2 , 4 , 3)$ ⟶vector form of the equation of straight line.

Note: t is a real number does not express a unique constant number but it takes different real values and it is called a parameter and for each value of this parameter(t), we can find a point on the straight line.

for example: when t = 1 , **then** $\vec{r} = (1, 3, 3)$ represents the position vector of a point on the straight line .

and when t = 2 , **then** $\vec{r} = (-1, 7, 6)$ represents the position vector of another point on the straight line.

Try to solve

2. Find the vector form of the equation of the straight line passing through point (4 , -2 , 5) and its direction vector is (1 , -2 , 2), then find another point on this straight line.

Learn

Parametric equations of a straight Line in space

From the vector equation of the straight line $\vec{r} = \vec{A} + t\,\vec{d}$

and by substituting for $\vec{r} = (x, y, z)$, $\vec{A} = (x_1, y_1, z_1)$, $\vec{d} = (a, b, c)$

then $(x, y, z) = (x_1, y_1, z_1) + t\,(a, b, c)$

$\therefore$ $(x = x_1 + at,\ y = y_1 + bt,\ z = z_1 + ct)$ ⟶ the parametric equations of the straight line

Example

3. Find the parametric equations for the straight line passing through point (2 , -1 , 3) and its direction vector is (4 , -2 , 5).

Solution

$\vec{r} = (2, -1, 3) + t\,(4, -2, 5)$ ⟶ **vector form of the equation of the straight line**

$\therefore (x, y, z) = (2, -1, 3) + t\,(4, -2, 5)$

$\therefore x = 2 + 4t \quad , \quad y = -1 - 2t \quad , \quad z = 3 + 5t$

Try to solve

3. Find the parametric equations for the straight line passing through the origin point and its direction vector is (-2 , 3, 1).

Learn

The cartesian equation of a straight Line in space

From the parametric equations of the straight line

$x = x_1 + t\,a,\ y = y_1 + tb,\ z = z_1 + t\,c$ $\therefore \dfrac{x - x_1}{a} = t,\ \dfrac{y - y_1}{b} = t,\ \dfrac{z - z_1}{c} = t$

$\therefore \dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$ ⟶ the cartesian form of the equation of the straight line.

Where each of a, b, c not equal to zero.

Note:

1- In the case a = 0 (say), then the cartesian form of the equation of the straight line takes the form of $x = x_1, \frac{y - y_1}{b} = \frac{z - z_1}{c}$

2- You learned in the previous years that the equation of the straight line in a plane is $a x + b y + c = 0$ and some people think that the equation of the straight line in space will be $a x + b y + c z + d = 0$ and that's a common mistake where the last equation represents the equation of a plane in space as we will see in the next lessons.

3- Since the direction ratios a , b , c are proportion to the direction cosine ℓ, m, n, then we can write the cartesian form of the equation of the straight line in the form

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Example

4) Find all the different forms of the equation of the straight line passing through the points (2 , -1 , 5) and (-3 , 1 , 4).

Solution

The direction vector of the straight line $\vec{d}$ = (-3 , 1 , 4) - (2 , -1 , 5) = (-5 , 2 , -1)

∴ **The vector form of the equation of the straight line** $\vec{r}$ = (2 , -1 , 5) + t (-5 , 2 , -1)

The parametric equations x = 2 - 5t , y = -1 + 2t , z = 5 - t

The Cartesian form $\frac{x - 2}{-5} = \frac{y + 1}{2} = \frac{z - 5}{-1}$

Try to solve

4) Find all the different forms of the equation of the straight line passing through point (3 , 2 , 0) and (-1 , 3 , 4)

Example

5) Find all the different forms of the equation of the straight line $\frac{3x + 1}{2} = \frac{y - 1}{2} = \frac{5 - z}{3}$

Solution

let $\frac{3x + 1}{2} = \frac{y - 1}{2} = \frac{5 - z}{3} = t$

∴ $\frac{3x + 1}{2} = t$, then $x = \frac{-1}{3} + \frac{2}{3}t$

$\frac{y - 1}{2} = t$, then $y = 1 + 2t$

$\frac{5 - z}{3} = t$, then $z = 5 - 3t$

The parametric form of the equation of the straight line

from the parametric equation, we can write the equation

$$(x , y , z) = (-\frac{1}{3} , 1 , 5) + t (\frac{2}{3} , 2 , -3)$$

i.e $\vec{r} = (-\frac{1}{3} , 1 , 5) + t (\frac{2}{3} , 2 , -3)$ vector form

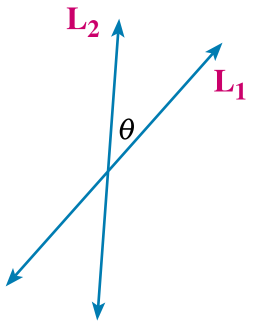
Notice that: the direction ratios of the straight line are $(\frac{2}{3} , 2 , -3)$ or (2 , 6 , -9)

Try to solve

(5) Find all the different forms of the equation of the straight line $\frac{x+4}{3} = \frac{2y+5}{2} = \frac{4-z}{4}$, then find a point belongs to the straight line.

Learn

The angle between two straight lines in space



If L_1 , L_2 are two straight lines in space whose direction vectors are $\overrightarrow{d_1} = (a_1, b_1, c_1)$ and $\overrightarrow{d_2} = (a_2, b_2, c_2)$, then the smallest angle between the two straight lines L_1, L_2 is given by the relation:

$$\cos\theta = \frac{|\overrightarrow{d_1} \cdot \overrightarrow{d_2}|}{\|\overrightarrow{d_1}\| \, \|\overrightarrow{d_2}\|}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

and if $(\ell_1, m_1, n_1), (\ell_2, m_2, n_2)$ are the direction cosines for the two straight lines, then:

$$\cos\theta = |\ell_1 \ell_2 + m_1 m_2 + n_1 n_2|$$

Example

(6) Find the measure of the angle between the two straight lines $\overrightarrow{r_1} = (2, -1, 3) + t_1(-2, 0, 2)$ and $x = 1$, $\frac{y-4}{3} = \frac{z+5}{-3}$

Solution

From the equation of the first straight line $\overrightarrow{d_1} = (-2, 0, 2)$

From the equations of the second straight line $\overrightarrow{d_2} = (0, 3, -3)$

$$\therefore \cos\theta = \frac{|\overrightarrow{d_1} \cdot \overrightarrow{d_2}|}{\|\overrightarrow{d_1}\| \, \|\overrightarrow{d_2}\|} = \frac{|(-2, 0, 2) \cdot (0, 3, -3)|}{\sqrt{(-2)^2 + 0^2 + 2^2}\sqrt{0^2 + 3^2 + (-3)^2}}$$

$$= \frac{6}{\sqrt{8}\sqrt{18}} = \frac{1}{2} \qquad \therefore \theta = 60^\circ$$

Try to solve

(6) Find the measure of the angle between the two straight lines
$L_1 : x = 2 - 5t, y = 1 - t, z = 3 + 4t$, $L_2 : \frac{x+1}{3} = \frac{2-y}{4} = \frac{z}{2}$

Example

(7) Find the measure of the angle between the two straight lines whose direction cosines are

$$\left(\frac{5}{13\sqrt{2}}, \frac{-12}{13\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Solution

$(\ell_1, m_1, n_1) = (\frac{5}{13\sqrt{2}}, \frac{-12}{13\sqrt{2}}, \frac{1}{\sqrt{2}})$ $(\ell_2, m_2, m_2) = (\frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}})$

$\therefore \cos\theta = |\ell_1\ell_2 + m_1m_2 + n_1n_2|$

$= |\frac{5}{13\sqrt{2}} \times \frac{-3}{5\sqrt{2}} + \frac{-12}{13\sqrt{2}} \times \frac{4}{5\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}|$

$= |\frac{-15}{130} + \frac{-48}{130} + \frac{1}{2}| = \frac{1}{65}$

$\therefore \theta = \cos^{-1}(\frac{1}{65}) = 89^\circ\ 7`\ 6``$

Try to solve

7 Find the measure of the angle between the two straight lines with direction cosines $(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$

Learn

Parallel lines in space

If $\overrightarrow{d_1} = (a_1, b_1, c_1)$, $\overrightarrow{d_2} = (a_2, b_2, c_2)$ are the direction vectors of the two straight lines L_1 and L_2, then $L_1 // L_2$ if and only if $\overrightarrow{d_1} // \overrightarrow{d_2}$. This condition can be satisfied by several forms

1- $\overrightarrow{d_1} = k\ \overrightarrow{d_2}$ **2-** $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ **3-** $\overrightarrow{d_1} \times \overrightarrow{d_2} = \overrightarrow{o}$

Note

1- If the two straight lines are parallel and there is a point on one of them satisfying the equation of the other, then the two straight lines are coincident .

2- If $\overrightarrow{d_1}$ is not parallel to $\overrightarrow{d_2}$, then L_1 and L_2 are either intersect or skew.

Example

8 Prove that the two straight lines $\overrightarrow{r_1} = \hat{j} + t_1(\hat{i} + 2\hat{j} - \overrightarrow{k})$

$\overrightarrow{r_2} = (\hat{i} + \hat{j} + \hat{k}) + t_2(-2\hat{i} - 2\hat{j})$

intersect at a point, then find their intersection point.

Solution

$\overrightarrow{d_1} = (1, 2, -1)$, $\overrightarrow{d_2} = (-2, -2, 0)$

$\because \frac{a_1}{a_2} = \frac{1}{-2} = -\frac{1}{2}$, $\frac{b_1}{b_2} = \frac{2}{-2} = -1$ $\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$\therefore$ The two straight lines are not parallel to prove that the two straight lines intersect at a point, look for a value for t_1, and t_2 which make $\overrightarrow{r_1} = \overrightarrow{r_2}$

$\therefore \hat{j} + t_1(\hat{i} + 2\hat{j} - \hat{k}) = \hat{i} + \hat{j} + \hat{k} + t_2(-2\hat{i} - 2\hat{j})$ by equating the cofficients

$\therefore t_1 = 1 - 2t_2$, then $t_1 + 2t_2 = 1$ **(1)**

$2t_1 = -2t_2$, then $t_1 + t_2 = 0$ **(2)**

$-t_1 = 1$, then $t_1 = -1$ **(3)**

by substitution from (3) in (1) $t_2 = 1$

This values satisfy equation (2)

∴ Thses two straight lines intersect at a point and the position vector of the point of the intersection is $\overrightarrow{r} = \hat{j} - 1(\hat{i} + 2\hat{j} - \hat{k}) = -\hat{i} - \hat{j} + \hat{k}$ **i.e** (-1 , -1 , 1)

Try to solve

8 Prove that the two straight lines $\overrightarrow{r_1} = (3, -3, 5) + t_1 (0, -5, 5)$

$\overrightarrow{r_2} = (-2, 3, 1) + t_2 (5, -1, -1)$

are perpendicular and intersect at a point, find the coordinates of their intersection point.

Learn

Perpendicular lines in space

If $\overrightarrow{d_1} = (a_1, b_1, c_1)$, $\overrightarrow{d_2} = (a_2, b_2, c_2)$ are the direction vectors of the two straight lines L_1 and L_2 , then

$L_1 \perp L_2$ if and only if $\overrightarrow{d_1} \cdot \overrightarrow{d_2} = 0$

Example

9 Prove that the two lines $\overrightarrow{r_1} = (1,2,4) + t_1 (2, -1, 1)$ & $\overrightarrow{r_2} = (1, 1, 1) + t_2(-2, 7, 11)$ are orthogonal, then show that they are skew.

Solution

$\overrightarrow{d_1} = (2, -1, 1)$ ⟶ direction vector of the first straight line

$\overrightarrow{d_2} = (-2, 7, 11)$ ⟶ direction vector of the second straight line

∴ $\overrightarrow{d_1} \cdot \overrightarrow{d_2} = (2, -1, 1) \cdot (-2, 7, 11)$

$= 2 \times (-2) + (-1) \times 7 + 1 \times 11$

$= -4 - 7 + 11$

$= 0$ ∴ the two straight lines are perpendicular

To prove that the two straight lines are skew, we prove that there are not any values for t_1, t_2 make $\overrightarrow{r_1} = \overrightarrow{r_2}$

i.e. (1 , 2 , 4) + ك$_1$ (2 , -1 , 1) = (1 , 1 , 1) + t_2 (-2 , 7 , 11) by equating the coefficients

∴ $1 + 2t_1 = 1 - 2t_2$, then $t_1 + t_2 = 0$ **(1)**

$2 - t_1 = 1 + 7t_2$, then $-t_1 - 7t_2 = -1$ **(2)**

$4 + t_1 = 1 + 11t_2$, then $t_1 - 11t_2 = -3$ **(3)**

By solving the two equations 1 and 2, we get $t_1 = \frac{-1}{6}$, $t_2 = \frac{1}{6}$ and these values do not satisfy the third equation

∴ the two straight lines are skew

Try to solve

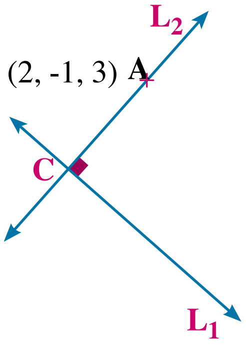
(9) Prove that the two straight lines

$\vec{r}_1 = (3, -1, 2) + t_1 (4, 1, 3)$, $\vec{r}_2 = (0, 4, -1) + t_2 (1, -1, 2)$ are skew.

Example

(10) Find the equation of the straight line passing through the point (2 , -1 , 3) and intersects the straight line $\vec{r}_1 = (1, -1, 2) + t (2, 2, -1)$ orthogonally.

Solution



Let C be the point of intersection of the two straight lines

$\because$ $C \in$ straight line L_1 (given straight line)

$\therefore$ C can be written in the form

C (1 + 2t , -1 + 2t , 2 - t)

the direction vector of L_2 (required straight line) is

$\vec{d}_2 = \overrightarrow{AC} = \vec{C} - \vec{A}$

$\therefore$ $\vec{d}_2 = (2t - 1, 2t, -t - 1)$

$\because$ $\vec{d}_1 = (2, 2, -1)$

$\because$ The two straight lines are perpendicular $\therefore \vec{d}_1 \cdot \vec{d}_2 = 0$

$\therefore$ $(2, 2, -1) \cdot (2t - 1, 2t, -t - 1) = 0$

$\therefore$ $4t - 2 + 4t + t + 1 = 0$

$\therefore$ $9t = 1$ then $t = \frac{1}{9}$

$\therefore$ $\vec{d}_2 = (\frac{-7}{9}, \frac{2}{9}, \frac{-10}{9}) = (-7, 2, -10)$

$\therefore$ the equation of L_2 is $\vec{r} = (2, -1, 3) + t_2 (-7, 2, -10)$

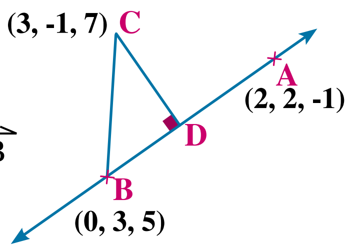
Try to solve

(10) Find the equation of the straight line passing through the origion point and intersects the straight line $\vec{r} = (3, 1, 4) + t (2, 1, 3)$ orthogonally.

Example (distance between a point and a straight line in space)

(11) Find the perpendicular distance from point (3 , -1 , 7) to the straight line passing through the two points (2 , 2 , -1) and (0 , 3 , 5)

Solution



Let A (2 , 2 , -1), B (0 , 3 , 5), C (3 , -1 , 7)

$\overrightarrow{BC} = \vec{C} - \vec{B} = (3, -1, 7) - (0, 3, 5) = (3, -4, 2)$

The direction vector of the straight line $\vec{d} = \overrightarrow{BA} = \vec{A} - \vec{B}$

$\therefore$ $\vec{d} = (2, -1, -6)$

B D is the absolute value of the projection $\overrightarrow{BC}$ on the straight line $\overleftrightarrow{AB} = \frac{|\overrightarrow{BC} \cdot \overrightarrow{BA}|}{\|\overrightarrow{BA}\|}$

$\therefore \quad BD = \frac{|(3, -4, 2) \cdot (2, -1, -6)|}{\sqrt{2^2+(-1)^2+(-6)^2}} = \frac{2}{\sqrt{41}}$

but $\|\overrightarrow{BC}\| = \sqrt{3^2 + (-4)^2 + 2^2} = \sqrt{29}$

$\therefore$ The perpendicular distance $CD = \sqrt{(BC)^2 - (DB)^2}$

$= \sqrt{29 - \frac{4}{41}} = \sqrt{\frac{1185}{41}} \simeq 5{,}3$ unit length

Try to solve

11) Find the length of the perpendicular drawn from point (2 , 1 , -4) to the straight line $\vec{r} = (1, -1, 2) + t\,(2, 3, -2)$

Critical thinking: Can you prove the following relation which identifies the distance from point B to the straight line $\vec{r} = \vec{A} + t\,\vec{d}$ the perpendicular distance $= \frac{\|\overrightarrow{AB} \times \vec{d}\|}{\|\vec{d}\|}$

Exercises 2 - 1

Complete:

1) The vector equation of the straight line passing through point (2 , -1 , 3) and the vector (-1 , 4 , 2) is a direction vector of it is ....

2) The measure of the angle between the two straight lines $2x = 3y = -z$ and $6x = -y = -4z$ equals

3) The measure of the angle between the two straight lines whose direction ratios are (1, 1, 2) and($\sqrt{3}$ -1, -$\sqrt{3}$ -1, 4) equals ....

4) If θ_z is the angle made by the straight line passing through point (3, -1 , 1) and the origin point with the +ve direction of z axis, then $\cos\theta_z$ = ...

5) A direction vector of the straight line passing through the two points (7 , -5 , 4)and (5 , -3 , 3) is..............................

Answer the following:

6) Find the direction cosines of the straight line with its direction ratios
   a) -1 , 2 , 3   b) 1 , 1 , 1

7) Find all the different forms of the equation of the straight line.
   a) Passes through point (4 , -2 , 5) and the vector $\vec{d} = (2, 1, -1)$ is a direction vector of it.
   b) Passes through point (3 , -1 , 5) and parallel to the vector $\overrightarrow{AB}$ where $\overrightarrow{AB} = (4, -2, 2)$

c) Passes through the two points (3 , -2 , 0) and (0 , 4 , 1)

d) Passes through point (3 , 2 , 5) and makes equal angles with the +ve directions of the coordinated axes.

8) Find the vector form of the equation of the straight line $x - 3 = \frac{y+2}{4} = \frac{2-z}{3}$

9) If $\overrightarrow{OA} = \hat{i} - 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = -\hat{i} - 3\hat{j}$,

$\overrightarrow{OC} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\overrightarrow{OD} = 8\hat{i} + \hat{j} + 4\hat{k}$.

Find the vector equation for each of the following straight lines

a) Passes through the two points A, B

b) Passes thorough point D and parallel to $\overleftrightarrow{BC}$

c) Passes through point C and intersects $\overleftrightarrow{AB}$ orthogonally

10) Find the measure of the angles between the two straight lines

a) L_1 : passing through the two points (-3 , 2 , 4) and (2 , 5 , -2)

L_2 : passing through the two points (1 , -2 , 2) and (4 , 2 , 3)

b) $L_1 : \vec{r} = (2 , -1 , 3) + t_1 (-1 , 4 , 2)$

$L_2 : \vec{r} = (0 , 2 , -1) + t_2 (1 , 1 , 3)$

c) $L_1 : 2x = 3y = 4z$

$L_2 : \frac{x-1}{3} = \frac{y+2}{-2} = \frac{z}{-5}$

11) State the necessary condition(s) to make the two straight lines

$L_1: x = x_1 + a_1t_1$, $y = y_1 + b_1t_1$, $z = z_1 + c_1t_1$

$L_2 : x = x_2 + a_2t_2$, $y = y_2 + b_2t_2$, $z = z_2 + c_2t_2$

a) parallel b) perpendicular c) intersect at a point

12) Find the vector equation of the straight line passing through point A (1 , -1 , 0) and parallel to the straight line passing through the two points B (-3 , 2, 1), C (2 , 1 , 0), then show that point D (-14 , 2 , 3) belongs to the straight line.

13) Find the value of n which makes the two straight lines $L_1 : \vec{r_1} = (3 , -1, n) + t_1 (4 , 1, 3)$

$L_2 : x = \frac{y-4}{-1} = \frac{z+1}{2}$ intersect at a point, then find the point of their intersection

14) **Discover the error:**

a) The sum of the squares of direction ratios for any straight line equals 1

b) The direction cosines of the straight line passing through the two points (x_1 , y_1 , z_1) , (x_2 , y_2 , z_2) is $(x_2 - x_1 , y_2 - y_1 , z_2 - z_1)$

c) If (a_1, b_1, c_1) and (a_2, b_2, c_2) are the direction ratios of the two straight lines L_1 and L_2 , then the measure of the angle between them is given by the relation $\cos\theta = |a_1a_2 + b_1b_2 + c_1c_2|$

Unit Two

2 - 2

The equation of a plane in space

You will learn

- Vector equation of a plane in space
- Standard equation of a plane in space
- General equation of a plane in space
- Angle between two planes
- Condition of parallel planes
- Condition of orthogonal planes
- Equation of intersection line of two planes in space
- The distance between a point and a plane
- The distance between two parallel planes

Key terms

- plane
- Standard form
- Parallel planes
- Perpendicular planes
- Interecting planes
- Angle

Materials

- Scientific calculator
- 3-D computer programmes

Think and discuss

1- If $\overrightarrow{A}$ and $\overrightarrow{B}$ two orthogonal vectors, then $\overrightarrow{A} \cdot \overrightarrow{B}$ = ...

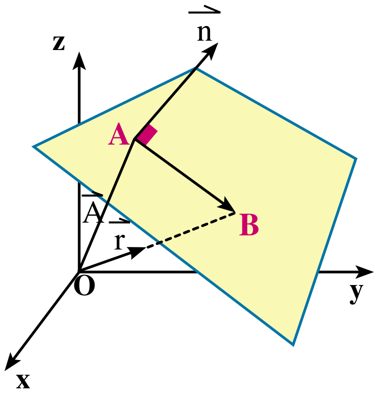
2- The direction vector of the straight line passing through the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is ....

3- The z -coordinate for all the points lying on x y-plane equals ....

Learn

The vector form of the equation of a plane in space

If point A (x_1, y_1, z_1) belongs to the plane and its position vector is $\overrightarrow{A}$ and the normal direction vector to the plane is $\overrightarrow{n} = (a, b, c)$ and B (x, y, z) any point on the plane its position vector is $\overrightarrow{r}$ then :



$$\overrightarrow{n} \cdot \overrightarrow{AB} = 0$$

$\therefore \quad \overrightarrow{n} \cdot (\overrightarrow{B} - \overrightarrow{A}) = 0 \qquad (\overrightarrow{B} = \overrightarrow{r})$

$\therefore \quad \overrightarrow{n} \cdot \overrightarrow{r} = \overrightarrow{n} \cdot \overrightarrow{A} \longrightarrow$ **the vector form of the equation of the plane.**

i.e: to find the vector equation of the plane, we must know a point on the plane and the perpendicular direction vector to the plane.

Example

1 Find the vector form of the equation of the plane that has normal vector $\overrightarrow{n} = \hat{i} + \hat{j} + \hat{k}$ and contains point (0 , 1, 1) .

Solution

The vector equation $\overrightarrow{n} \cdot \overrightarrow{r} = \overrightarrow{n} \cdot \overrightarrow{A}$ where $\overrightarrow{A} = (0 , 1 , 1)$

$\therefore \quad (1 , 1 , 1) \cdot \overrightarrow{r} = (1 , 1 , 1) \cdot (0 , 1 , 1)$

$(1 , 1 , 1) \cdot \overrightarrow{r} = 2$

Try to solve

1) Find the vector form of the equation of the plane passing through point (2 , -3 , 1) and the vector $\overrightarrow{n} = (1 , -2 , 3)$ is normal to the plane.

Learn

The standard form and general form of the equation of a plane in space

From the vector form of the equation of the plane

$$\overrightarrow{n} \cdot (\overrightarrow{r} - \overrightarrow{A}) = 0$$

where $\overrightarrow{n} = (a , b , c), \ \overrightarrow{r} = (x , y , z), \ \overrightarrow{A} = (x_1 , y_1 , z_1)$

$\therefore (a , b , c) \cdot (x - x_1 , y - y_1 , z - z_1) = 0$

$\therefore a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \longrightarrow$ **the standard form for the equation of the plane**

by expanding

$\therefore ax + by + cz + (-ax_1 - by_1 - cz_1) = 0$

consuming $-ax_1 - by_1 - cz_1 = d$ **, then**

$ax + by + cz + d = 0 \longrightarrow$ **the general form of the equation of the plane**

Example

2) Find the standard form and the general form of the equation of the plane passing through point (3 , -5 , 2) and the vector $\overrightarrow{n} = (2 , 1, 1)$ is normal to the plane.

Solution

The standard form $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\therefore \ 2(x - 3) + (y + 5) + (z - 2) = 0 \longrightarrow$ **the standard form**

Expand and collect the like terms

$\therefore \ 2x + y + z - 3 = 0 \longrightarrow$ **general form**

Try to solve

2) Find all the different forms of the equation of the plane passing through point (-3 , 4 , 2) and the vector $\overrightarrow{n} = (1 , -1, 3)$ is normal to the plane.

Example **(Equation of a plane passing through three non-colinear points)**

3) Find the different forms of the equation of the plane passing through points (3 ,-1,0), (2 ,1 ,4) and (0 , 3 , 3).

Solution

First : we must make sure that the points are non-colinear

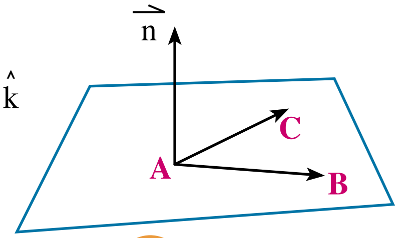
Assuming A (3 , -1 , 0), B (2 , 1 , 4), C (0 , 3 , 3)

$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = (-1, 2, 4)$, $\overrightarrow{AC} = \overrightarrow{C} - \overrightarrow{A} = (-3 , 4 , 3)$

$\because \frac{-1}{-3} \neq \frac{1}{2}$ $\therefore \overrightarrow{AB} \nparallel \overrightarrow{AC}$ $\therefore$ the points are non-colinear

To find the equation of the plane, we need the normal vector to the plane by getting the vector product of the two vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$.

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 4 \\ -3 & 4 & 3 \end{vmatrix} = -10\,\hat{i} - 9\,\hat{j} + 2\,\hat{k}$$



$\therefore$ **The vector form of the equation of the plane**

$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{A}$

$\therefore$ $(-10 , -9 , 2) \cdot \vec{r} = (-10 , -9 , 2) \cdot (3 , -1, 0)$

$\therefore$ $(-10 , -9 , 2) \cdot \vec{r} = -21$

The standard form of the equation of the plane

$a (x - x_1) + b (y - y_1) + c (z - z_1) = 0$

$\therefore$ $-10 (x - 3) - 9 (y + 1) + 2z = 0$

The general form of the equation of the plane

$(-10 , -9 , 2) \cdot (x , y, z) = -21$

$\therefore$ $-10x - 9y + 2 z + 21 = 0$

Tip

The equation of the plane passing through the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Try to solve

3. Find all the different forms of the equation of the plane passing through points (1 , 0, 0), (0 , 2, 0) and (0 , 0 , 3).

Example (a plane contains two straight lines)

4. Prove that the two straight lines $\vec{r_1} = (3\,\hat{i} + \hat{j} - \hat{k}) + t_1 (\hat{i} + 2\,\hat{j} + 3\,\hat{k})$

$\vec{r_2} = (2\,\hat{i} + 5\,\hat{j}) + t_2 (\hat{i} - \hat{j} + \hat{k})$

are intersecting , then find the equation of the plane containing them.

Solution

If the two straight lines intersect, then $r_1 = r_2$

$\therefore$ $(3\,\hat{i} + \hat{j} - \hat{k}) + t_1 (\hat{i} + 2\,\hat{j} + 3\,\hat{k}) = (2\,\hat{i} + 5\,\hat{j}) + t_2 (\hat{i} - \hat{j} + \hat{k})$

by equating the coefficients ,we find

$3 + t_1 = 2 + t_2$, then $t_1 - t_2 = -1$ **(1)**

$1 + 2t1 = 5 - t_2$, then $2t_1 + t_2 = 4$ **(2)**

$-1 + 3t_1 = t_2$, then $3t_1 - t_2 = 1$ **(3)**

by solving 1 , 2 $t_1 = 1$, $t_2 = 2$

by substituting with these values in the equation (3), we find it satisfies the equation

$\therefore$ The two straight lines intersect.

the normal vector to the plane is $\overrightarrow{n}$ where

$$\overrightarrow{n} = \overrightarrow{d_1} \times \overrightarrow{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 5\ \hat{i} + 2\ \hat{j} - 3\ \hat{k}$$

The vector equation of the plane $\overrightarrow{n} \bullet \overrightarrow{r} = \overrightarrow{n} \bullet \overrightarrow{A}$

$\therefore (5 , 2 , -3) \bullet \overrightarrow{r} = (5 , 2 , -3) \bullet (3 , 1 , -1)$

$\therefore (5 , 2 , -3) \bullet \overrightarrow{r} = 20$

The general form

$(5 , 2 , -3) \bullet (x , y , z) = 20$

$\therefore 5x + 2y - 3z = 20$

Try to solve

4. Prove that the two straight lines $L_1 : 2x = 3y = 4z$, $L_2 : 3x = 2y = 5z$ are intersecting, then find the equation of the plane containing the two straight lines .

Example

5. Find the point of intersection of the straight line $2x = 3y - 1 = z - 4$ with the plane $3x + y - 2z = 5$

Solution

From the equation of the plane

$$y = 5 + 2z - 3x$$

by substituting in the straight line equation

$$2x = 14 + 6z - 9x$$
$$= z - 4$$

$$11x - 6z = 14 \quad (1)$$

$$-5z + 9x = 18 \quad (2)$$

by solving the equations (1) , (2), we get

$$x = -38 , z = -72$$

by substituting in the plane equation

$$\therefore y = -25$$

$\therefore$ the point of intersection is $(-38 , -25 , -72)$

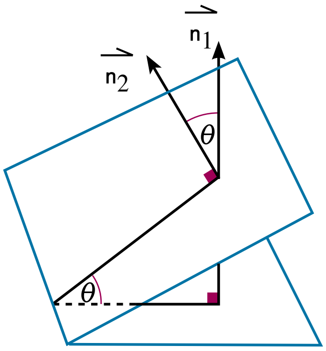
Try to solve

5. Find the intersection point of the straight line $\overrightarrow{r} = (1 , 4 , 2) + t\,(3 , 2, 2)$ with the plane $(3 , 2, 2) . \overrightarrow{r} = -2$

Learn

the angle between two planes

The measure of the angle between two planes is the measure of the angle between their two normal vectors i.e. $\vec{n_1}$ and $\vec{n_2}$ are the two normal vectors on the two planes, then the measure of the angle between the two planes is given by the relation



$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{\|\vec{n_1}\| \, \|\vec{n_2}\|} \quad \text{where } 0 \leqslant \theta \leqslant 90^\circ$$

Example

6 Find the measure of the angle between the two planes P_1: $(2, -1, 4) \cdot \vec{r} = 5$ and P_2: $3x - y + 2z = 4$

Solution

The normal vector to the first plane $\vec{n_1} = (2, -1, 4)$

the normal vector to the second plane $\vec{n_2} = (3, -1, 2)$

$\therefore$ The measure angle between the two planes is θ where

$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{\|\vec{n_1}\| \, \|\vec{n_2}\|} = \frac{|(2, -1, 4) \cdot (3, -1, 2)|}{\sqrt{2^2 + (-1)^2 + 4^2}\ \sqrt{3^2 + (-1)^2 + 2^2}} = \frac{15}{7\sqrt{6}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{15}{7\sqrt{6}}\right) = 28^\circ 58'$$

Try to solve

6 Find the measure of the angle between the two planes $x - 3y + 2z = 0$ and $2x + y - z = 3$

Parallel planes and perpendicular planes

If $\vec{n_1}$ and $\vec{n_2}$ are the normal vectors to the two planes, then

1- The two planes are parallel if $\vec{n_1} \,/\!/\, \vec{n_2}$ i.e $\left(\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}\right)$

2- the planes are perpendicular if $\vec{n_1} \cdot \vec{n_2} = 0$ i.e. $(a_1a_2 + b_1b_2 + c_1c_2) = 0$

Example

7 If the plane $2x - y + kz = 5$ is parallel to the plane $x + Ly + 4z = 1$, find the value of k, L.

Solution

$\because$ The two plane are parallel $\quad \therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\therefore \frac{2}{1} = \frac{-1}{L} = \frac{k}{4} \quad \therefore L = \frac{-1}{2}, \ k = 8$

Try to solve

7 If the plane $x - 3y + z = 4$ is perpendicular to the plane $ax + 2y + 3z = 2$, find the value of a

Example **(The equation of the line of intersection of two planes)**

8 Find the equation of the line of intersection of the two planes $x + 2y - 2z = 1$, $2x + y - 3z = 5$

Solution

by eliminating x from the two equations by multiplying the first equation by -2 and adding to the second

$\therefore \quad -3y + z = 3$, then $z = 3y + 3$ **(1)**

by eliminating y from the two equations by multiplying the second equation by -2 and adding to the first

$\therefore \quad -3x + 4z = -9$, then $z = \frac{3x - 9}{4}$

$\therefore \quad \boxed{\frac{3x - 9}{4} = \frac{3y + 3}{1} = \frac{z}{1}}$ **the equation of the line of intersection**

Another solution:

$x + 2y - 2z = 1$ **(1)**

$2x + y - 3z = 5$ **(2)**

by eleminating x

$-3y + z = 3$ **(3)**

let z = k

(3) $y = \frac{k - 3}{3}$, **(2)** $x = \frac{9 + 4k}{3}$

$\therefore$ **the parametric equations of the line of intersection are**

$x = 3 + \frac{4}{3}k$, $y = -1 + \frac{1}{3}k$, $z = k$

Third solution:

The line of intersection is perpendicular to the two normal vectors $\vec{n_1}$, $\vec{n_2}$.

$\therefore$ The direction vector of the intersection line $\vec{d}$ can be calculated using the vector product of the two vectors $\vec{n_1}$, $\vec{n_2}$

$$\vec{d} = \vec{n_1} \bullet \vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 2 & 1 & -3 \end{vmatrix} = -4\hat{i} - \hat{j} - 3\hat{k}$$

To find a point on the intersection line, put $x = 1$

By substituting in the equation of the first plane $2y - 2z = 0$ **(1)**

By substituting in the equation of the second plane $y - 3z = 3$ **(2)**

By solving equations (1), (2), we get $z = -\frac{3}{2}$, $y = -\frac{3}{2}$

$\therefore$ The point $(1 , -\frac{3}{2} , -\frac{3}{2})$ lies on the line of intersection .

The equation of the intersection line $\vec{r} = (1 , -\frac{3}{2} , -\frac{3}{2}) + t(-4 , -1 , -3)$

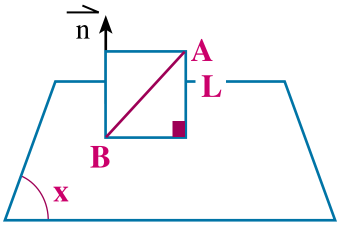
Try to solve

8 Find the equation of the intersection line of the two planes $3x - y + 2z = 3$ and $x - 2y + 5z = 2$

Learn

the length of the perpendicular drawn from a point to a plane

If A (x_1, y_1, z_1) is a point outside the plane x and B is a point on the plane , $\vec{n}$ is the normal vector to the plane, then the distance from the point A to the plane equals the length of the projection of $\vec{BA}$ to $\vec{n}$



$$L = \frac{|\vec{BA} \cdot \vec{n}|}{\|\vec{n}\|}$$

Example

9 Find the length of the perpendicular drawn from the point (1 , -1 , 3) to the plane whose equation is $\vec{r} \cdot (2, 2, -1) = 5$

Solution

We must get a point on the plane and the normal vector to the plane $\vec{r} \cdot (2, 2, -1) = 5$ then $\vec{n} = (2, 2, -1)$

To find a point on the plane, we assume that the plane cuts z-axis at the point z (0 , 0 , z)

$\therefore$ $(0, 0, z) \cdot (2, 2, -1) = 5$, **then** $z = -5$

$\therefore$ The point B (0 , 0 , -5) lies on the plane

$\vec{BA} = \vec{A} - \vec{B} = (1, -1, 8)$ where A (1 , -1 , 3)

the length of the perpendicular $(L) = \frac{|\vec{BA} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|(1, -1, 8) \cdot (2, 2, -1)|}{\sqrt{2^2 + 2^2 + (-1)^2}} = \frac{8}{3}$ units

Try to solve

9 Find the distance between point (-2 , 1 , 4) and the plane whose equation $\vec{r} \cdot (1, -3, 2) = 4$

Cartisian form of the perpendicular length drawn from a point and a plane

You notice that the perpendicular length from point A (x_1, y_1, z_1) to the plane passing through point b (x_2, y_2, z_2) and $\vec{n} = (a, b, c)$ is the normal vector to the plane is given by the relation

$$L = \frac{|\vec{BA} \cdot \vec{n}|}{\|\vec{n}\|}$$

$$\therefore L = \frac{|(x_1 - x_2, y_1 - y_2, z_1 - z_2) \cdot (a, b, c)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_1 + b y_1 + c z_1 + (-a x_2 - b y_2 - cz_2)|}{\sqrt{a^2+b^2+c^2}}$$

$\because$ point B (x_2, y_2, z_2) lies on the plane $a x + b y + c z + d = 0$

$\therefore$ $-a x_2 - b y_2 - c z_2 = d$

$\therefore$ $L = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$ **the Cartesian form of the length of the perpendicular**

Example

(10) Find the length of the perpendicular drawn from point (1 , 5, -4) to the plane whose equation $3x - y + 2z = 6$

Solution

$$L = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|3(1) - (5) + 2(-4) - 6|}{\sqrt{3^2 + (-1)^2 + 2^2}} = \frac{16}{\sqrt{14}} \text{ unit length}$$

Try to solve

(10) Find the length of the perpendicular drawn from point (-1 , 4 , 0) to the plane with equation $x - 2y - z = 4$

Example **(The distance between two parallel planes)**

(11) Prove that the two planes $x + 3y - 4z = 3$, $2x + 6y - 8z = 4$ are parallel, then find the distance between them.

Solution

To prove that the palanes are parallel, we prove that their normal vectors are parallel.

$\vec{n_1} = (1, 3, -4)$, $\vec{n_2} = (2, 6, -8)$

$\because$ $\frac{a_1}{a_2} = \frac{1}{2}$ $\qquad$ $\frac{b_1}{b_2} = \frac{3}{6} = \frac{1}{2}$, $\qquad$ $\frac{c_1}{c_2} = \frac{-4}{-8} = \frac{1}{2}$

$\therefore$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ $\qquad$ $\therefore$ The two planes are parallel

To find the distance between them, we find a point on one of them, then we find the distance between this point and the other plane.

To get a point on the first plane, we assume $x = 0$, $y = 0$

by substituting in the first plane equation

$\therefore z = \frac{-3}{4}$

$\therefore$ point $(0, 0, \frac{-3}{4})$ lies on the first plane

Then the perpendicular length from this point to the second plane is

$$L = \frac{|2(0) + 6(0) - 8(\frac{-3}{4}) - 4|}{\sqrt{2^2 + 6^2 + (-8)^2}} = \frac{\sqrt{26}}{14} \text{ unit length}$$

Try to solve

11) Prove that the two planes $3x + 6y + 6z = 4$, $x + 2y + 2z = 1$ are parallel and find the distance between them.

Learn

Equation of a plane using the intercepted parts from the coordinate axes

If a plane cuts the coordinates axes at points $(x_1 , 0 , 0)$, $(0 , y_1 , 0)$ $(0 , 0 , z_1)$, then the equation of the plane is in the form

$$\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$$ ⟶ **the equation of the plane interms of the intercepted parts from the coordinate axes**

Ask your teacher to prove the previous form of the plane equation.

Example

12) Find the equation of the plane which intercepts the coordinate axes x, y, z the parts 2, -3, 0 respectively.

Solution

The equation of the plane is $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$

i.e. $\frac{x}{2} + \frac{y}{-3} + \frac{z}{5} = 1$

Try to solve

12) Find the intercepted parts made by the plane $2x + 3y - z = 6$ with the coordinate axes.

Critical thinking:

If the plane $3x + 2y + 4z = 12$ intersects the coordinate axes x , y , z at the points a , b , c respectively, find the area of the triangle A B C

Exercises 2 - 2

Choose the correct answer :

1) Which of the following points belongs to the plane $2x + 3y - z = 5$

a (1 , 1 , 1) b (1 , 2 , 0) c (0 , 2 , 1) d (3 , 2 , -1)

2) The plane $3x - 2y + 4z = 12$ intercepts from x -axis a part of length

a 3 b -4 c 4 d 6

3) If the intercepted parts from the coordinate axes by the plane $x + 5y - 6z = 30$ are a , b , c, then a + b + c =

a 0 b 30 c 31 d 41

4) The equation of the plane which passes through the point (1 , 2 , 3) and parallel to the coordinate axes x , y is

a) $x + y = 3$ b) $z = 3$ c) $x = 1$ d) $y = 2$

5) Then equation of the plane passing through points (2 , 3 , 5) , (-1 , 3 , 1) , (4 , 3 , -2) is

a) $x + y - z = 0$ b) $x = -1$ c) $y = 3$ d) $z = -2$

6) The equation of the plane passing through point (1 , -2 , 5) and the vector (2 , 1 , 3) is perpendicular to it is

a) $2x + y + 3z = 1$ b) $2x + y + 3z = 15$

c) $x - 2y + 5z = 15$ d) $x + y + z = 4$

Answer the following:

7) Find all the different forms of the equation of the plane passing through point (1 , -1 , 4) and its normal vector is $\vec{n} = (2 , -3 , 4)$, then show the following:

a) Is point (2 , 2 , 1) lying on the plane?

b) Is the vector $\vec{u} = (3 , -5 , -2)$ parallel to the plane?

8) Find three points in space belonging to each of the following planes:

a) $x = 3$ b) $y = -2$ c) $x + 3y = 5$ d) $2x - y + 3z = 4$

9) Find the general equation of the plane passing through the origin point and the vector $\vec{n} = \hat{i} + 2\hat{j} - 3\hat{k}$ is normal to it.

10) Find all the different forms of the equation of the plane passing through point (2 , -1 , 0) and the vector $\vec{n} = 4\hat{i} + 10\hat{j} - 7\hat{k}$ is normal to it.

11) Find all the different forms of the equation of the plane passing through the three points a (2 , -1 , 0) b (-1 , 3 , 4) , c (3 , 0 , 2)

12) Prove that the straight line $\vec{r} = \hat{k} + t(2\hat{i} + 3\hat{j} + 4\hat{k})$ is perpendicular to the plane $x + \frac{3}{2}y + 2z = 5$

13) Prove that point A (2 , 3 , 1) & the straight line L: $\vec{r} = (3\hat{i} + \hat{j} + 3\hat{k}) + t(\hat{i} - 2\hat{j} + 2\hat{k})$ lying on the plane whose equation is $\vec{r} \cdot (2\hat{i} - \hat{k}) = 3$

14) Find the equation of the plane passing through point (2 , 1 , 4) and satisfies the following condition:

a) Parallel to the plane $2x + 3y + 5z = 1$

b) Perpendicular to the straight line passing through the two points (3 , 2 , 5) , (1 , 6 , 4)

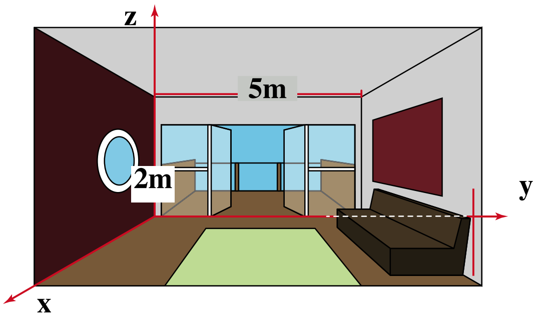
c) Perpendicular to each of the planes $7x + y + 2z = 6$, $3x + 5y - 6z = 8$

15) Find the coordinates of the point of intersection of the straight line $\vec{r} = \hat{k} + t(2\hat{i} + \hat{j} + \hat{k})$ with the plane $\vec{r} \bullet \hat{i} = 4$

16 Find all the different forms of the equation of the plane that intercepts 2 , 4 , 5 from the coordinate axes x , y , z respectively.

17 **Environment:** In the opposite figure, find the equation for each of:

- **a** The floor plane.
- **b** The cieling plane.
- **c** The lateral walls planes.



18 Find the equation of the plane which contains the straight line
L_1: $\vec{r} = (0 , 3 , -5) + t_1 (6 , -2 , -1)$ and parallel to the straight line
L_2 : $\vec{r} = (1 , 7 , -4) + t_2 (1 , -3 , 3)$

19 Find the measure of the angle between the following pairs of planes:

- **a** $P_1 : 2x - y + z = 5$, P_2: $3x + 2y -2$ع $= 1$
- **b** P_1: $\vec{r} \cdot (2 , 1 , -1) = 4$, P_2: $\vec{r} \cdot (3 , -2 , 0) = 7$
- **c** $P_1 : y = 4$, P_2: $x - 3y + 5z = 1$

Questions of multi-requirements

20 If the points A , B , C , D are in space whose position vectors are $-\hat{j} + \hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$, $-\vec{x} - 2\vec{y} + 2\vec{z}$, $7\vec{x} - 4\vec{y} + 2\vec{z}$ respectively

- **a** Find the normal vector to the plane ABC
- **b** Show that the length of the perpendicular from D to the plane A B C equals $2\sqrt{6}$
- **c** Show that the two planes A B C , D B C are orthogonal.
- **d** Find the equation of the line of the intersection of the two planes A B C , O D B

21 If the plane X contains points A (1 , 4 , 2) B (1 , 0 , 5), C (0 , 8 , -1) and the plane Y contains point D (2 , 2 , 3) and the vector $\vec{n} = \hat{j} + 2\hat{j} + 2\hat{k}$ is perpendicular to it, find:

- **a** The cartisian equation of X
- **b** The cartisian equation of the plane Y
- **c** What is the values of t , f if point (t , 0 , f) belongs to each of the two planes X , Y?
- **d** Find the vector equation of the line of intersection of the two planes X , Y
- **e** If point (1 , 1, p) equidistant from the two planes X , Y, find all possible values of p.

Unit summary

Direction vector:

1. If ℓ, m, n are the direction cosines of a straight line, then the vectors $\overrightarrow{d} = t\,(\ell, m, n)$ represents the direction vector of the straight line and is denoted by $\overrightarrow{d} = (a, b, c)$ where (a, b, c) are called the direction rations of the straight line.

2. the direction vector of the straight line takes different equivalent forms such as $\overrightarrow{d} = 2\,(\ell, m, n) = 3(\ell\,, m, n) = -4\,(\ell, m, n)$ ....

Equation of the straight line

- The equation of the straight line which passes through point (x_1, y_1, z_1) and the vector $\overrightarrow{d} = (a, b, c)$ is directed vector

  the vector form: $\overrightarrow{r} = (x_1, y_1, z_1) + t\,(a, b, c)$
- **The parametric equations:** $x = x_1 + t\,a\,,\; y = y_1 + t\,b,\; z = z_1 + t\,c$
- **The cartesian equation:** $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

The angle between two straight lines

If $\overrightarrow{d_1}$ and $\overrightarrow{d_2}$ are the direction vectors of two straight lines, then the smallest angle between the two straight lines is:

$$\cos\theta = \frac{|\overrightarrow{d_1} \cdot \overrightarrow{d_2}|}{\|\overrightarrow{d_1}\|\,\|\overrightarrow{d_2}\|}$$

and if (ℓ_1, m_1, n_1) and (ℓ_2, m_2, n_2) are the direction cosines of the two straight lines, then:

$$\cos\theta = |\ell_1, \ell_2 + m_1\,m_2 + n_1\,n_2|$$

The parallelism and perpendicularity conditions of two straight lines

If $\overrightarrow{d_1} = (a_1, b_1, c_1)$ $\overrightarrow{d_2} = (a_2, b_2, c_2)$ are the direction vectors of two straight lines, then

- **The two straight lines are parallel if** :

$$\overrightarrow{d_1} = k\,\overrightarrow{d_2} \quad , \quad \text{or} \quad \overrightarrow{d_1} \times \overrightarrow{d_2} = \overrightarrow{o} \quad , \quad \text{or} \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

- **The two straight line are orthogonal if:**

$$\overrightarrow{d_1} \cdot \overrightarrow{d_2} = 0 \text{ or } a_1\,a_2 + b_1\,b_2 + c_1\,c = 0$$

The equation of a plane

The equation of the plane passing through point (x_1, y_1, z_1) and the vector $\vec{n} = (a, b, c)$ is perpendicular to the plane .

- **Vector form :** $\vec{n} \bullet \vec{r} = \vec{n} \bullet (x_1, y_1, z_1)$
- **Standard form:** $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
- **General form :** $a x + b y + c z + d = 0$

The angle between two planes

- If $\vec{n_1} = (a_1, b_1, c_1)$, $\vec{n_2} = (a_2, b_2, c_2)$ are the normal vectors to the planes, then the measure of the angle between the two planes is given by the relation :

$$\cos\theta = \frac{|\vec{n_1} \bullet \vec{n_2}|}{\|\vec{n_1}\| \, \|\vec{n_2}\|} \qquad \text{where } 0 \leqslant \theta \leqslant 90^\circ$$

Parallel and orthogonal planes

- If $\vec{n_1}$, $\vec{n_2}$ are the perpendicular vectors to the two planes, then the condition of parallelism of the two planes is

$$\vec{n_1} \,//\, \vec{n_2} \quad \text{or} \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

- the condition of perpendicularity of the two planes is

$$\vec{n_1} \cdot \vec{n_2} = 0 \quad \text{or} \quad a_1 \ a_2 + b_1 b_2 + c_1 c_2 = 0$$

The perpendicular length drawn from a point and a plane

- Length of the perpendicular drawn from $a(x_1, y_1, z_1)$ to the plane passes through $B(x_2, y_2, z_2)$ and vector $\vec{n} = (a, b, c)$ is perpendicular to the plane.

$$L = \frac{|\vec{BA} \bullet \vec{n}|}{\|\vec{n}\|} \qquad \text{Vector form}$$

or

$$L = \frac{|ax_1 + b y_1 + c z_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \qquad \text{cartesian form}$$

General exercise

Choose the correct answer:

1. The equation of the straight line passing through point A(-1, 0, 2) and the vector $\vec{d} = (1, -1, 3)$ is a direction vector of it is

 a) $\frac{x-1}{2} = \frac{y}{1} = \frac{z-1}{-1}$ b) $\frac{x+1}{1} = \frac{y}{-1} = \frac{z-2}{3}$

 c) $\frac{x-1}{3} = \frac{y}{-1} = \frac{z}{1}$ d) $\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z}{2}$

2. The equation of the straight line passing through points A(1, -1, 2) , B (-1, 0, 1) is

 a) $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-2}{1}$ b) $\frac{x+1}{-2} = \frac{y}{1} = \frac{z-1}{-1}$

 c) $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z-1}{2}$ d) $\frac{x-1}{1} = \frac{y+1}{3} = \frac{z-2}{1}$

3. The measure of the angle between the two straight lines $\frac{x-3}{2} = \frac{z+1}{-2}$, y = 1and $\frac{x+1}{1} = \frac{y-2}{2} = \frac{z+1}{-2}$ equals

 a) 15° b) 30° c) 45° d) 60°

4. If the two straight lines L_1: $\frac{x}{2} = \frac{y-1}{-1} = \frac{z-2}{m}$, L_2 : $\frac{x-1}{m} = \frac{y-2}{1} = \frac{z}{-1}$ are perpendicular. What is the value of m?

 a) -1 b) 2 c) 1 d) -3

5. If the straight lines L_1: x = 2t - 1, y = t + 1, z = t - 1, L_2: x = at - 1, y = 2t + 1, z = b t - 2 are parallel, then a + b =

 a) 4 b) -2 c) 6 d) -2

6. The point (2, -1, 3) lies on the plane

 a) x + y - z = 6 b) 2x - 3y + z = -10 c) 3x - 2y + 4z = 20 d) x - 2y + 5z = 4

7. If the plane $\frac{x}{4} + \frac{y}{2} + \frac{z}{2} = 1$ cuts the coordinate axes at points A, B, C, then the area of the triangle A B C =

 a) 12 b) 10 c) 6 d) 4

8. The length of the perpendicular from point (2, 3, 1) to the plane 2x - 2y + z = 5 is

 a) 1 b) 2 c) 3 d) 4

9. The equation of the line of intersection of the two planes L_1: 2x - y + z - 1 = 0, L_2: x - 3y - z + 2 = 0 is

 a) $\frac{x+1}{-1} = \frac{y}{2} = \frac{z}{3}$ b) $\frac{x-1}{1} = -\frac{y}{3} = \frac{z-5}{1}$

 c) $\frac{x-2}{1} = \frac{y-3}{-2} = \frac{z}{-1}$ d) $\frac{x-1}{4} = \frac{y-1}{3} = \frac{z}{-5}$

10 The two straight lines $L_1: \frac{x+1}{-1} = \frac{y-2}{-1} = \frac{z-1}{3}$, $L_2: \frac{x+1}{1} = \frac{y-2}{-2} = \frac{z-1}{-1}$ lie on the plane

a $3x - 5y + z - 1 = 0$ b $5x - 4y + 2z - 7 = 0$

c $7x - 5y - z - 4 = 0$ d $7x + 2y + 3z = 0$

Answer the following question:

11 Find the distance between point (-2, 4, -5) and the straight line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

12 Find the distance between point (2, 1, -1) and the plane $\vec{r} \cdot (\hat{i} - 2\hat{j} + 4\hat{k}) = 9$

13 Find the coordinates of the point of intersection the straight line passing through the two points (3, -4, -5) (2, -3, 1) with plane passing through points (2, 2, 1) , (3, 0, 1) , (4, -1, 0)

14 Find the coordinates of the point of intersection of the straight line $\vec{r} = (2, -1, 2) + t\,(3, 4, 2)$ with plane $\vec{r} \cdot (1, -1, 1) = 5$

15 Find the projection of point A(0, 9, 6) on the straight line passing through the two points B (1, 2, 3) and C (7, -2, 5)

16 Prove that the two planes $2x + y + 2z = 8$, $4x + 2y + 4z + 5 = 0$ are parallel, then find the distance between them.

Creative thinking

17 If a plane intersects the coordinate axes at the points A, B, C and the point (p, q, r) is the point of intersection of the medians of triangle A B C,

Prove that the equation of the plane is $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$

Accumulative test

Complete the following:

1 The measure of the angle between the straight line $\frac{x-1}{\sqrt{2}} = \frac{y-\sqrt{2}}{1} = \frac{z+1}{1}$ with the +ve direction of z axis is ....

2 the length of the perpendicular drawn from the point (-1, 0 , 1) to the straight line $\frac{x-1}{2} = \frac{y-1}{1} = \frac{z+1}{-1}$ equals ....

3 The parametric equations for the straight line passing through the two points A(-1, 0, 3), B(1, -1, 0) is ....

4 The measure of the angle between the two planes
$-x - y + \sqrt{2}\,z + 1 = 0$, $x - y + \sqrt{2}\,z - 3 = 0$ equals ....

5 The equation of the plane passing through point (2, 3, -1) and the vector $\overrightarrow{n} = (5, 2, -3)$ is perpendicular to it is ....

6 The plane $3x - 4y + z + 10 =$ zero cuts from y-axis a part of length ....

7 The point of intersection of the straight line $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{3}$ and the plane $x - 2y + 3z + 5 = 0$ is ....

Choose the correct answer from the following:

8 The distance between point (a, b, c) and the x -axis is

a) $\sqrt{a^2 + c^2}$ b) $\sqrt{a^2 + b^2}$ c) $\sqrt{b^2 + c^2}$ d) $\sqrt{b^2 + c^2}$

9 The equation of the x-axis in space is

a) $x = 0 , y = 0$ b) $x = 0 , z = 0$ c) $y = 0 , z = 0$ d) $x = 0$

10 The equation of the straight line passing through the two points (2, -1, 3) , (0, 3, 1)

a) $\overrightarrow{r} = (2, -1, 3) + t(2, -4, 2)$ b) $\overrightarrow{r} = (2, -1, 3) + t(2, 2, 4)$

c) $\overrightarrow{r} = (2, -4, 2) + t(2, -1, 3)$ d) $\overrightarrow{r} \cdot (2, -4, 2) = 0$

11 The point lying on the straight line $\overrightarrow{r} = (2, -1, 3) + t\,(1, 2, -1)$ is:

a) (1, 1, 1) b) (0, 2, -2) c) (3, 1, 2) d) (4, -3, 0)

12 The distance between the two planes $y = 4$, $y = -2$ is

a) 3 units b) two units c) 6 units d) 8 units

Answer the following questions:

13 Write the cartesian equation for each of the following straight lines:

a) $\overrightarrow{r} = (1, 3, 9) + t\,(5, 4, 2)$

b) The straight line passing through the point (0, 2, 0) and the vector $\overrightarrow{d} = (3, -1, 4)$ is a direction vector of it

14 Find the measure of the angle between

a) The two straight lines $L_1 : 2x = 3y - 1 = z - 3$, $L_2: \overrightarrow{r} = (2, -1, 5) + t\,(-1, 1, 2)$

b) The two planes $3x - y = 5$, $x - 2y = 4$

15 Find the cartesian equation of the plane $(x, y, z) = (2, 3, 5) + t_1\,(-1, 3, 4) + t_2\,(6, 1, -2)$ where t_1, t_2 are parameters

16 Find the measure of the angle between the two planes $2x + 2y + 7z = 8$

$3x - 4y + 4z = 5$

For more activities and exercises, visit www.sec3mathematics.com.eg

First test

First : Answer one of the following questions

First question: Choose the correct answer:

1. If ${}^{n}C_{n-3} = 20$, then n = ... (a) 3 (b) 4 (c) 5 (d) 6)
2. $i + i^2 + i^3 + \ldots\ldots + i^{100} =$ (a) 0 (b) 1 (c) 2 (d) 100)
3. If A (7, -1, 8) , B (11, 2, -4), then $\overline{AB}$ = ...cm
   (a) 10 (b) 11 (c) 12 (d) 13)
4. $x^2 + y^2 + z^2 + 4x - 6y + 8z + 4 =$ is an equation of a sphere , whose diameter length = ...cm
   (a) 5 (b) 10 (c) 15 (d) 20)
5. If $L_1 \ \frac{x-3}{1} = \frac{y+2}{-2} = \frac{z+1}{-4}$ is parallel to $L_2 : \frac{x+5}{-2} = \frac{y}{k+1} = \frac{z-1}{8}$, then k = ...
   (a) 3 (b) 4 (c) 5 (d) 6)
6. If θ is the measure of the angle included between the two vectors $\vec{A} = (-2, -6, 1)$, $\vec{B} = (2, 6, -1)$, then $\theta =$ ....
   (a) 30° (b) 60° (c) 120° (d) 180°)

Second question: Complete each of the following:

1. The coefficient of x^5 in the expansion of $(3 - 2x)^7$ equals ....
2. The solution set of $\begin{vmatrix} x & 1 & 2 \\ 0 & x & 3 \\ 0 & 0 & x \end{vmatrix} - 8 = 0$ in R is ....
3. If $\vec{A} = 2\hat{i} + 3\hat{j} + m\hat{k}$, $\vec{B} = -6\hat{i} - 4\hat{j} + 4\hat{k}$ and $\vec{A} \perp \vec{B}$, then m = ...
4. If $\vec{A} = (3, 0, 4)$, $\vec{B} = \hat{j} - 2\hat{j} + 3\hat{k}$, then $\vec{A} \times \vec{B} =$ ....
5. The equation of the sphere whose center is (2, -3, 1) and its radius length equals $2\sqrt{5}$ is ....
6. The equation of the straight line passing through the two points A(2,-1,4), B(-1,0,2) is ....

Answer the following question:

Questions three:

1. In the expansion of $(2x + \frac{1}{x^2})^{15}$, find the value of the term free of x and prove that this expansion does not contain a term includes x^5
2. Find all the different forms of the equation of ther straight $\frac{x+3}{2} = \frac{2y-1}{5} = \frac{3z+2}{4}$

Question four:

1. Find the multiplicative inverse of the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 1 \\ 1 & 5 & 21 \end{pmatrix}$

2. Find the two square roots of the complex number $z = 2 - 2\sqrt{3}\ i$ in the trigonometric form.

Fifth Question:

1. Solve the following equations $x + 3y + 2z = 13$, $2x - y + z = 3$, $3x + y - z = 2$ using the multiplicative inverse of the matrix

2. Find the point of intersection of the planes $2x + y - z = -1$, $x + y + z - 2 = 0$, $3x - y - z = 6$

Second Test

First : Answer one of the following:

First question: Choose the correct answer:

1. If the two equations $2x + y = 1$, $4x + 2y = k$ have an infinite number of solutions, then k = ...

(a zero   b 1   c 2   d 3)

2. If $^{n+1}C_3 : {}^{n}C_4 = 2 : 3$, then n = ... (a 2   b 3   c 5   d 11)

3. If $x^2 + y^2 + z^2 + 6x - 4y + 10z - 8 = 0$ is the equation of a circle whose centre is M, then M = ...

a (-3 , 2, -5)   b (4, -2, -5)   c (-3 , -2, -5)   d (3 , 2 , 5)

4. If $\overrightarrow{A} = (-2, 4, 6)$, $\overrightarrow{B} = (0, k, 3)$ where $k \in Z^+$ and $\| \overrightarrow{AB} \| = 7$, then the value of k = ...

a 10   b 8   c 6   d 4

5. If θ is the measure of the angle included between $\overrightarrow{A} = (2, 0, 2)$, $\overrightarrow{B} = (0, 0, 4)$, then θ = ...

a 30°   b 45°   c 60°   d 90°

6. If $L_1 : \frac{x-3}{2} = \frac{-y-1}{6} = \frac{z}{k}$ is parallel to $L_2 : \frac{x+2}{6} = \frac{y-4}{M} = \frac{z-1}{3}$, then K + M = ...

a -17   b -10   c 10   d 17

Second question: Complete

1. $\omega + \omega^2 + \ldots\ldots\ldots\ldots\ \omega^{100} =$ ....

2. If a, b, c are the lengths of the sides of the triangle ABC then the value of $\begin{vmatrix} a & b & c \\ 5 & 7 & 8 \\ \sin A & \sin B & \sin C \end{vmatrix}$ = ...

3. If $\overrightarrow{A} = (-1, 4, 2)$, $\overrightarrow{B} = (2, 2, 1)$, then the component of $\overrightarrow{A}$ in the direction of $\overrightarrow{B}$ = ...

4. $x^2 + y^2 + z^2 - 4k x + 4y - 8z + 2k = 0$ is the equation of a sphere , the length of its radius equals $2\sqrt{5}$ then the value of k = ...

5. If the two planes $3x - y + 2z + 3 = 0$, and $k x - 4y + z - 5 = 0$ are perpendicular , then the value of k = ...

6. If C (-1, 6, -5) is the midpoint of $\overline{AB}$ where A (k - 2, -1, m + 3), b (2, n - 7, -2), then k + m - n = ...

Second : Answer the following question:

Third question:

1. Find the coefficient of x5 in the expansion of $(1 - x + x^2)(1 + x)^{11}$
2. Prove that the straight line $\frac{x - 1}{2} = \frac{y + 3}{-1} = \frac{z}{3}$ intersects the plane $3x + 2y + z - 8 = 0$ at a point and find the measure of the inclination angle of the line with the plane.

Fourth question:

1. Calculate the rank of the matrix $\begin{pmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 3 & -5 & 2 \end{pmatrix}$

   hence prove that the equations $2x - y - 3z = 2$, $x + 2y + z = 1$, $3x - 5y + 2z = 13$ have a unique solution and find this solution using the multiplicative inverse of the matrix.

2. Find the exponential form of the complex number $z = \frac{2 + 6i}{3 - i}$, then find z^{-1} , $\overline{z}$ ، $\sqrt{z}$ in the trigonometric form .

Fifth question:

1. Prove that one of the values of the expression $\sqrt{i} - \sqrt{-i} = \sqrt{2}$
2. If $(x - 2)^2 + (y + 4)^2 + (z - 2)^2 = 1$, $(x + 4)^2 + (y - 4)^2 + (z - 2)^2 = 4$ are the equations of two spheres , find the distance between the centres of the two spheres and show that the two spheres do not intersect

The third test

First : Answer one of the following questions:

First question : choose the correct answer

1. The sum of the coefficients of the expansion of $(1 + x)^5$ equals

   a) zero b) 5 c) 32 d) $\lfloor 5$

2. If x is a complex number , then the number of solutions of the equation $\begin{vmatrix} x^3 + 1 & x - 1 \\ x + 1 & x^3 - 1 \end{vmatrix} = 0$ equals

   a) 6 b) 5 c) 4 d) 3

3. If (x, y, z) is the midpoint of $\overline{AB}$ where A(-4, 0, 5), B (-2, 4, -13), then x + y + z = ...

   a) -5 b) -6 c) 3 d) 4

4. If A (-4, -2, 3) , B (1, 2, k) and the length of $\overline{AB} = \sqrt{77}$, then one of the values of k is ....

   a) 2 b) 4 c) 6 d) 9

5. If $\vec{A} = (-1, 3, 4)$, $\vec{B}$ (0, -2, 5) , then $\|\vec{AB}\| =$

   a) $2\sqrt{3}$ b) $3\sqrt{3}$ c) $4\sqrt{3}$ d) $5\sqrt{3}$

6. The length of the perpendicular drawn from point A(3, 0, -5) on the plane $2x + \sqrt{5}\,y + 4z - 6 = 0$ equals ....

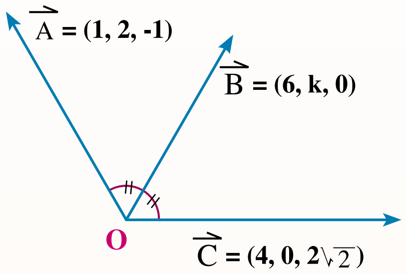
a) 4 b) 5 c) 6 d) 7

Question 2: Complete each of the following:

1. If $z = \sin 60^\circ - i \cos 60^\circ$, then the principle amplitude of z = ...

2. The rank of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 5 & 5 & 5 \end{pmatrix}$ = ...

3. In the opposite figure, the value of k = ...



4. The radius length of the sphere $x^2 + y^2 + z^2 + 4x - 6y + 8z + 4 = 0$ equals ....

5. If the straight line $\frac{x+3}{2} = \frac{y+1}{-6} = \frac{z-2}{k}$ is parallel to the straight line $\frac{x+2}{4} = \frac{y-5}{m} = \frac{z-1}{3}$, then k + m = ...

6. If the straight line $\frac{x+2}{6} = \frac{y-1}{m} = \frac{z-1}{3}$ is perpendicular to the straight line $\frac{x-9}{-2} = \frac{y+8}{1}$, then m = ...

Answer the following question:

Third question:

1. If $(m + x)^n = 3a + 6ax + 5ax^2 + \ldots\ldots$ where $n \in Z^+$, find the value of each of m and a

2. Prove that the following system of equations has a solution except the non zero solution and write the general form of these solutions

$$2x - y + 3z = 0 \quad , \quad 4x + 5y - z = 0 \quad , \quad 2x + 3y - z = 0$$

Fourth Question:

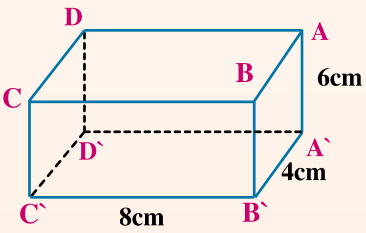
1. If $|z_1| = |z_2| = 1$, and the org $(z_1 z_2^3) = 81^\circ$, org $(\frac{z_1}{z_2}) = 33^\circ$, write in the form of x + y i the number $(z_1^{15} + z_2^{15})$

2. Find the length of the perpendicular drawn from point A (-2, 3, 1) to the line $\frac{x+2}{2} = \frac{y-3}{4} = \frac{z-1}{4}$

Fifth question:

1. Prove that $\begin{vmatrix} a_1 & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$

2. in the opposite figure , A B C D C`B`C`D is a cuboid, find $\overrightarrow{BD`} \cdot \overrightarrow{CA`}$



The fourth test

First : Answer one of the following questions:

First Question: Choose the correct answer:

1. If ${}^{10}C_{r+1} : {}^{10}C_{r-1} = 21 : 10$, then the value of $r =$ ....
   - a) 3
   - b) 4
   - c) 5
   - d) 6

2. If $\begin{vmatrix} \log_2 3 & 3 & 9 \\ 0 & \log_3 5 & 7 \\ 0 & 0 & \log_5 x \end{vmatrix} = 4$, then $x =$ ....
   - a) 16
   - b) 32
   - c) 64
   - d) 128

3. If $\overrightarrow{A} = (1, -1, 2)$, $\overrightarrow{B} = (0, 2, -3)$, $\overrightarrow{C} = (-2, 1, 0)$, then $\|3\overrightarrow{A} - \overrightarrow{B} + \overrightarrow{C}\| =$ ....
   - a) $8\sqrt{3}$
   - b) 11
   - c) 12
   - d) $7\sqrt{2}$

4. If $L_1: \frac{x+2}{-1} = \frac{y+3}{3} = \frac{z+5}{2}$ is perpendicular to $L_2: \frac{x}{2} = \frac{y-5}{k} = \frac{z-6}{m}$, then $3k + 2m =$ ....
   - a) -1
   - b) 0
   - c) 2
   - d) 4

5. The measure of the angle between the two straight lines $x - 1 = \frac{y+2}{\sqrt{2}} = -z + 1$, $-x = z + 3$, $y = 4$ equals ....
   - a) 45°
   - b) 120°
   - c) 135°
   - d) 150°

6. The direction cosines of the vector (2, -4, 4) are
   - a) (2, -4, 4)
   - b) (1, -2, 2)
   - c) $(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3})$
   - d) $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$

Second questions: complete:

1. $(3 + 7\omega + 3\omega^2)(3 - 7\omega^2 + 3\omega) =$ ....
2. The rank of the matrix $A = \begin{pmatrix} 2 & -6 \\ -3 & 3 \\ 4 & -12 \end{pmatrix}$ equals ....
3. The centre of the sphere $x^2 + y^2 + z^2 + 8x - 12y + 2z + 1 = 0$ equals ....
4. A B C D is a square of side length 10 cm, then $\overrightarrow{AB} \cdot \overrightarrow{AC} =$ ....
5. The unit vector in the direction of $\overrightarrow{A} = (2, 3, 2\sqrt{3})$ equals ....
6. The length of the perpendicular drawn from point (-2, -3, 1) to x-axis equals ....

Answer the following questions:

Third question:

1. Find the greatest term in the expansion of $(3 + 2x)^6$ at $x = 1$
2. Find the volume of a parallelepiped in which three adjacent sides are represented by the vectors:
   $\vec{A} = (1, -1, 2)$, $\vec{B} = (3, -2, 0)$, $\vec{C} = (0, 2, 4)$

Fourth question:

1. Find the roots of the equation $Z^4 + 4 = 0$ in the trigonometric form
2. If $\vec{A}$, $\vec{B}$, $\vec{C}$ are three mutually perpendicular unit vectors
   find: (a) $\|2\vec{A} - \vec{B} + 3\vec{C}\|$ (b) If $\vec{A} = (\frac{1}{3}, \frac{-2}{3}, \frac{2}{3})$, $\vec{B} = (\frac{3}{5}, 0, \frac{-4}{5})$ find $\vec{C}$

Fifth question:

1. Discuss the possibility for solving the following equations and write this solution, if exists: $x + y = 2,\ 2x + 3y = 5$
2. If $z = \sin\frac{\pi}{9} + i\cos\frac{\pi}{9}$, find $(\bar{z})$ in the trigonometric form and find the cubic roots of the number $(\bar{z})^9$

The fifth test

First : Answer one of the following questions:

First question: Choose the correct answer :

1. If $36\ {}^{2n-1}P_{n-1} = 9\ {}^{2n}P_n$, then n = ...
   (a) 1 (b) 2 (c) 3 (d) 4)
2. If the two equation $x + y = 2$, $2x + ky = 4$ has more than one solution, then k = ...
   (a) -2 (b) -1 (c) 1 (d) 2)
3. If $\overrightarrow{AB} = -3\hat{i} + 3\hat{j} + 7\hat{k}$, $\overrightarrow{BC} = \hat{j} + 5\hat{k}$, then $\|\overrightarrow{AC}\|$ = ...
   (a) 13 (b) 12 (c) 10 (d) 9)
4. If $\vec{A} = (-7, 3, 10)$ $\vec{B} = (-4, -1, -2)$, then the unit vector in the direction of $\overrightarrow{AB}$ = ...
   (a) $(\frac{3}{13}, \frac{4}{13}, \frac{12}{13})$ (b) $(\frac{3}{13}, \frac{-4}{13}, \frac{-12}{13})$ (c) $(\frac{-3}{13}, \frac{4}{13}, \frac{12}{13})$ (d) $(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13})$
5. If $\vec{A} = (1, -1, 2)$ $\vec{B} = (3, -2, 0)$, $\vec{C} = (0, 2, 4)$, then $\vec{A} \cdot \vec{B} \times \vec{C}$ = ...
   (a) 10 (b) 12 (c) 14 (d) 16
6. The length of the perpendicular drawn from point A(1, 0, 2) to the straight line $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-3}{-2}$ equals ....
   (a) $\frac{\sqrt{26}}{4}$ (b) $\frac{\sqrt{26}}{5}$ (c) $\frac{\sqrt{26}}{3}$ (d) $\frac{\sqrt{26}}{6}$

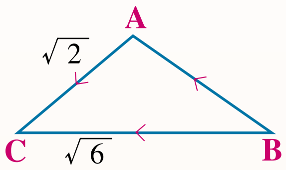
Second Question: Complete:

1) $(2 + \frac{3}{\omega})(2 + \frac{3}{\omega^2})(3 - \frac{2}{\omega})(3 - \frac{2}{\omega^2}) = ...$

2) If the coefficients of T_6 , T_{16} in the expansion of $(a + b)^n$ are equal, then n = ...

3) Cosine the measure of the angle between the two lines:
$\frac{x}{1} = \frac{y}{-2} = \frac{z+1}{-2}$ and , $\frac{x}{1} = \frac{y-2}{-2} = \frac{z}{2}$ equals ....

4) In the opposite figure, If $\| \overrightarrow{BC} \| = \sqrt{6}$, $\| \overrightarrow{AC} \| = \sqrt{2}$, $\overrightarrow{BA} = (-1, 0, 1)$
, then $\overrightarrow{BA} \bullet \overrightarrow{BC}$ = ...



5) The general equation of the sphere whose centre is (3, 4, -5) and touches y z plane is ....

6) The vectors form of the equation of the straight line which passes through point (2, -1, 4) and its direction vector is $\overrightarrow{d}$ = (4, 7, 1) is ....

Answer the following questions:

Third question:

1) In the expansion of $(1 + x)^{18}$ according to the ascending powers of x, If the coefficients of T_{2r+4} , T_{r-2} a are equal, find the value of r.

2) If the length of the perpendicular drawn from point A(0, -1, 2) to the plane $\sqrt{2}\,x + y - z + k = 0$ equals 2 unit length , find the value of k.

Fourth question:

1) Solve the following equations 2x + y - 2z = 10 , x + 2y + 2z = 1, 5x + 4y + 3z = 6
using the multiplicative inverse of the matrix

2) If $Z_1 = \frac{6 + 4i}{1 + i}$, $Z_2 = \frac{26}{5 - i}$ If $Z = 4\,(Z_1 - Z_2)$, find the cubic roots of z in the exponential form

Fifth question:

1) Without expanding the determinant , prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & cb \\ ac & cb & c^2+1 \end{vmatrix} = a^2 + b^2 + c^2 + 1$

2) If the plane 2x - y - z + 12 = 0 the sphere $(x + 3)^2 + (y + 2)^2 + (z - 1)^2 = 15$, find the area of the cross section (trace)

The sixth test

First : Answer one of the following:

First question: Choose the correct answer:

1. If ${}^{n}C_3 : {}^{n-1}C_4 = 8 : 5$, then the value of n ....
   (a) 5 (b) 7 (c) 8 (d) 9

2. The coefficient of the middle term in the expansion of $(3x - \frac{1}{6})^{10}$ equals ....
   (a) $\frac{-63}{8}$ (b) $\frac{-67}{8}$ (c) $\frac{63}{8}$ (d) $\frac{67}{8}$

3. The measure of the angle included between the two planes: $x + y - 1 = 0$, $y + z - 1 = 0$ equlas
   (a) 30 (b) 45 (c) 60 (d) 75

4. If $\overrightarrow{A} = (2, 1, -2)$, $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{C} \times \overrightarrow{B}$, then $\overrightarrow{B}$ = ...
   (a) (2, -1, -2) (b) (2, 1, -2) (c) (-2, -1, 2) (d) (-2, -1, 3)

5. If A(-2, 0, 3) , B(4, 2, -5), then $\| \overrightarrow{AB} \|$ = ... length unit
   (a) $\sqrt{12}$ (b) $\sqrt{40}$ (c) $\sqrt{44}$ (d) $\sqrt{104}$

6. If $\overrightarrow{A} \perp \overrightarrow{B}$, $\overrightarrow{A} \perp \overrightarrow{C}$, $\overrightarrow{B} = (2, 3, 2)$, $\overrightarrow{C} = (1, 2, 1)$ and $\| \overrightarrow{A} \| = 4\sqrt{2}$, then $\overrightarrow{A}$ = ...
   (a) (2, 3, 1) (b) 6(-4, 0, 4) (c) (4, 4, 0) (d) (0, -4, 4)

Second question: Complete:

1. $(1 - \frac{1}{\omega})(1 - \frac{1}{\omega^2})(1 - \frac{1}{\omega^4})(1 - \frac{1}{\omega^8})$ .... to 10 factors = ...

2. The rank of the matrix $\begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ equals ....

3. The direction vector of the straight line $\frac{x+2}{3} = \frac{z-1}{2}$ equals ....

4. If the measure of the angle between the two lines $\frac{x}{C} = \frac{y}{2} = \frac{z}{1}$, $\frac{x}{2} = \frac{y}{1} = \frac{z}{-1}$ equals 60°, then the value of a = ...

5. If A (1, 0, 0) and B (1, 1) lie on the plane $k x + y + m z + 2 = 0$, then k + m = ...

6. If $\overrightarrow{A} = (1, 0, 2)$, $\overrightarrow{B} = (2, -1, -2)$, then $(\overrightarrow{A} \times \overrightarrow{B}) \bullet (\overrightarrow{B} \times \overrightarrow{A})$ = ...

Answer the wollowing questions:

Third question:

1. If the coefficients of the fourth , fifth and sixth terms in the expansion $(2x + y)^n$ according to the descending powers of x form an arithmetic sequence , find the value of n

2. A sphere of centre (1, 2, 1) touches the plane $x + y + z = 1$, find the equation of the sphere

Fourth question:

1. Discuss the possibility of solving the set of the following system equations: $4x + 3y - 5z = 6$, $3x + 2y + 4z = 12$, $5x - 2y - 7z = 1$, then find the solution set of these equations using the multiplicative inverse

2. If $Z_1 = (\frac{\sqrt{3} + i}{2})^4$, $Z_2 = \sin\frac{\pi}{3} + i\cos\frac{\pi}{3}$, $i^2 = -1$ and $z = \frac{z_1}{z_2}$, find the square roots of z in the trigonometric form

Fifth question:

1. Without expanding the deterimant , prove that $\begin{vmatrix} x & a & b \\ a & x & b \\ b & a & x \end{vmatrix} = (x + a + b)(x - a)(x - b)$

2. Find the different forms of the equation of the straight line passing through (2, 1, -3) and parallel to the straight line $\frac{x - 1}{5} = \frac{y + 3}{2} = \frac{1 - z}{3}$

The seventh test

First Answer one of the following:

First question: Choose the correct answer:

1. If ${}^{30}C_r = {}^{30}C_{r + 10}$, ${}^{n}P_7 = 90 \times {}^{n-2}P_5$, then $|\underline{n - r}$ =

   a) zero b) 1 c) 10 d) $|\underline{20}$

2. If the equations $3x - 2y + z = 0$, $6x - 5y + 2z = 0$, $9x - 6y + kz = 0$ have solutions other than then zero solution , then k =

   a) zero b) 1 c) 3 d) 4

3. The length of the perpendicular drawn between the two planes $3x + 12y - 4z = 9$, $3x + 12y - 4z = -17$ equals

   a) 2 b) 3 c) 4 d) 5

4. If $\overrightarrow{A} = (4, -k, 6)$, $\overrightarrow{B} = (2, 2, m)$ and $\overrightarrow{A} // \overrightarrow{B}$, then k + m =

   a) -3 b) -2 c) -1 d) zero

5. If the straight line $x = 3y = az$ is parallel to the plane $x + 3y + 2z + 4 = 0$, then a =

   a) 3 b) 2 c) 1 d) -1

6) If $\overrightarrow{A} = (1, -2, 1)$, $\overrightarrow{B} = (-2, 1, 2)$, then the component of $\overrightarrow{A}$ in the direction of $\overrightarrow{B}$ =

a) $(\frac{4}{9}, \frac{-2}{9}, \frac{-4}{9})$ b) $(\frac{4}{9}, \frac{2}{9}, \frac{4}{9})$ c) $(\frac{-4}{9}, \frac{-2}{9}, \frac{-2}{9})$ d) $(\frac{4}{9}, \frac{-2}{9}, \frac{-4}{9})$

Second question: Complete:

1) $(\frac{3 + 5\omega}{5 + 3\omega^2} + \frac{5 + 3\omega^2}{3 + 5\omega})^8 =$ ....

2) the rank of the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ 1 & -3 & 4 \end{pmatrix}$ equals ....

3) If the plane X: $x - z + 1 = 0$, and the plane Y : $2x - 2y - z = 0$, then the measure of the angle between the two planes = ... °

4) The radius length of the sphere $(x - 2)^2 + (y + 4)^2 + (z - 5)^2 = 64$ equals ....

5) If $\overrightarrow{A} = (4, -5, 1)$, $\overrightarrow{B} = (2, -k, -2)$, $\overrightarrow{C} = (-4, 4, m - 2)$ and $\overrightarrow{AB} \parallel \overrightarrow{C}$, then $k + m =$ ....

6) If $\|\overrightarrow{A}\| = 2$, $\|\overrightarrow{B}\| = 3$, $\|\overrightarrow{C}\| = 12$ and $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$ are mutually orthogonal , then $\|\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}\| =$ ....

Answer the following questions:

Third question

1) If $Z_1 = (\sin\frac{\pi}{9} + i\cos\frac{\pi}{9})^5$, $Z_2 = (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^4$ and $z = \frac{z_1}{z_2}$, find the square roots of z in its exponontial form

2) If $\overrightarrow{A} = (2\cos\theta , \log_3 x, \sin\theta)$, $\overleftarrow{B} = (\cos\theta , \log_3 27 , 2\sin\theta)$ and $\overleftarrow{A} \cdot \overleftarrow{B} = 11$, find the value of x

Fourth question

1) In the expansion of $(1 + x)^n$ according to the ascending power of x if $T_3 = 17$, $3\, T_2 \times T_4 = 544$, find the value for each of n and x

2) without expanding the determinant, prove that

$$\begin{vmatrix} a + b + 2 & a & b \\ 1 & 2a + b + 1 & b \\ 1 & a & a + 2b + 1 \end{vmatrix} = 2(a + b + 1)^3$$

Fifth question:

1) If $A = \begin{pmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{pmatrix}$ and $A^T = A^{-1}$, find the value for each of x, y, z

2) Find the point of intersection of the straight line $x = y = z$ and the plane $x + 2y + 3z = 12$

The eighth test

First: Answer one of the following questions

First question: Complete:

1. If $\underline{|1 + \log x} = 1$, then x = ... or ....

2. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ a & b & c \end{vmatrix} = 5$, then the value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ a+5 & b+5 & c+5 \end{vmatrix}$ = ...

3. The measure of the angle between the two lines $\vec{r_1} = (-2, 5, -7) + k\,(-6, 6, 8)$, $\vec{r_2} = (1, -2, 3) + k\`\,(4, 12, -6)$ equals ....

4. If $\|\vec{A}\| = 4$, $\|\vec{B}\| = 6$ and the measure of the angle between the two vectors $\vec{A}$, $\vec{B}$ equals 60°, then $(2\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B})$ = ...

5. The equation of the sphere whose diameter is $\overline{AB}$ where A (7, 1, -4), B (3, -1, 2) is ....

6. If $\vec{A} = (1, 2, -4)$, $\vec{B} = (1, 1, k - 1)$ and $\|\vec{A} + \vec{B}\| = 7$ unit of length, then k = ...

Second question: Choose the correct answer

1. If $\frac{a^2 + b^2}{a + bi} = 2 + 3i$, then $a \times b$ = where a, b $\in R^*$

   a) -6 b) -5 c) 5 d) 6

2. The rank of the matrix $A = \begin{pmatrix} 0 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{pmatrix}$ equals

   a) 3 b) 2 c) 1 d) zero

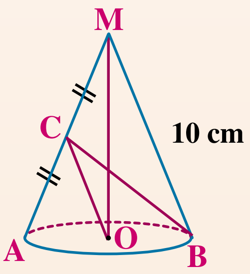
3. A B C D is a parallelogram in which $\vec{AB} = (2, 2, -1)$, $\vec{AD} = (-1, 2, -3)$, then the surface area of the parallelogram = ... cm^2

   a) 6 b) $7\sqrt{2}$ c) $3\sqrt{11}$ d) $\sqrt{101}$

4. In the opposite figure, a right circular cone, the perimeter of its base = 12π cm,

   C is the midpoint of $\vec{AM}$, then $\vec{BC} \cdot \vec{CO}$ =

   a) -43 b) -40
   c) -37 d) -33



5. If $\overrightarrow{A} = \hat{i} + \hat{j} + \hat{k}$, $\overrightarrow{B} = 2\hat{i} - \hat{j} - \hat{k}$, then $\overrightarrow{A} \times (\overrightarrow{A} - \overrightarrow{B}) =$

a $\hat{i} + \hat{k}$ b $-3\hat{j} + 3\hat{k}$ c $-3\hat{i} - 3\hat{j}$ d $3\hat{i} - 2\hat{j}$

6. If $L_1 : x = 0 , y = z , L_2 : y = 0 , x = z$ are two straight lines in space, the measure of the angle between them is θ ,then $\theta =$

a 45° b 120° c 150° d 165°

Answer the following questions:

Third question:

1. Use the multiplicative inverse of a matrix to solve the following equations:
$2x - y + z = -1, x - z = 2 , x + y = 3$

2. Find the point of intersection of the planes $2x + y - z = -1, x + y + z = 2, 3x - y - z = 6$

Fourth Question:

1. If $z_1 = 1 - \sqrt{3}\, i$, $z_2 = \cos\theta + i \sin\theta$, $z_3 = (\cos\frac{\theta}{2} - i\sin\frac{\theta}{2})^2$ and $z = \frac{z_1 z_2}{z_3}$, find the modulus and the principle amplitude of z , then find the square roots of z in its trigonometric form when $\theta = \frac{\pi}{6}$

2. Discuss the possibility of existence of a solution except the zero solution for the system of linear equations :
$x + 3y - 2z = 0 , x - 8y + 8z = 0 , 3x - 2y + 4z = 0$

Fifth question:

1. In the expansion of $(x^2 + \frac{1}{2x})^{3n}$ according to the descending powers of x :

First: Prove that the term free of x is of order $(2n + 1)$

Second: find the ratio between the term free of x and the middle term when $n = 4, x = 1$

2. If the two spheres $(x - 3)^2 + y^2 + (z - 3)^2 = 16, (x + 1)^2 + (y - 4)^2 + (z - k)^2 = 25$ are tangential, find the value of k

The ninth test

First : Answer one of the following:

First question: Complete:

1. If ${}^{x+y}P_4 = 360$, $\underline{|2x+y} = 5040$, then ${}^{y}C_{2x} =$ ....

2. The solution set of the equation $\begin{vmatrix} a+1 & 3 & 2 \\ 0 & a-1 & 5 \\ 0 & 0 & 7 \end{vmatrix} = 21$ is ....

3. Cosine the angle between the two vectors $\overrightarrow{A} = (1, -3, 0)$, $\overrightarrow{B} = (2, 0, 1)$ equals ....

4. The radius length of the sphere: $x^2 + y^2 + z^2 + 2x - 2y - 4z - 3 = 0$ equals ....

5. If $\overrightarrow{A} = (\frac{-1}{2}, \frac{3}{4}, k)$ is a unit vector, then the value of $k =$ .... or ....

6. If $\overrightarrow{A} = (k, -3, 1)$, $\overrightarrow{B} = (2, 3, -k)$ are perpendicular, then the value of $k =$ ....

Second question: Complete:

1. $(1 + \omega)^4 + (1 + \omega^2)^4 + (\omega + \omega^2)^4 =$ ....

2. The rank of the matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ equals ....

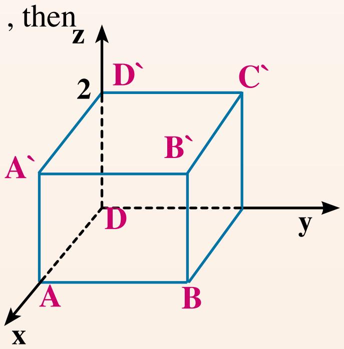
3. If $\overrightarrow{A} = (3, -2, k)$, $\overrightarrow{B} = (i, m, 2)$ and $\overrightarrow{A} \,//\, \overrightarrow{B}$, then $K =$ .... , $m =$ ....

4. If the measure of the angle which $\overrightarrow{C} = (2, 4, k)$ makes with the positive direction of y-axis equals 45° , then $k =$ ....

5. If the two planes: $x + 2y + kz = 2$, $3x - y + 2z + 4 = 0$ are perpendicular , then $k =$ ....

6. In the opposite figure, A B C D A`B C D is a cube of side length unity , then

   $\overrightarrow{AB`} \cdot \overrightarrow{BD} =$ ....



Answer the following question

Third question:

1. If $z_1 = 2 (\sin \frac{\pi}{3} + i \cos \frac{\pi}{3})$, $z_2 = \sqrt{2} (\sin \frac{\pi}{4} - i \cos \frac{\pi}{4})$, $z_3 = 1 + \sqrt{3}\, i$

   Find the number $z = \frac{z_1^3 \times z_2^4}{z_3^5}$ in its exponential form, then find the square roots of z in its trigonometric form

2. If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the midpoint of the line segment joining the centres of the two spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$, $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$, find the value of a.

Fourth Question:

1. Use the multiplicative inverse of a matrix to solve the following equations :
   $x - 2y + 2z = 2$, $3x + 4z = 10$, $6z - y = 5$

2. Prove that the term free of x in the expansion of $(x^2 + \frac{1}{x^3})^{5n}$ where $n \in Z^+$ equals $\frac{\lfloor 5n}{\lfloor 2n \ \lfloor 3n}$

Fifth question:

1. Find the value of k which makes the equations: $kx + y + z = 1$, $x + ky + z = 1$
   $x + y + kz = 1$ have an infinite number of solutions.

2. Find the length of the perpendicular drawn from point (-4 , 1 , 1) on the line $\frac{x+3}{1} = \frac{y-1}{\sqrt{5}} = \frac{z+2}{2}$

The tenth test

First : Answer one of the following:

B

First question: Complete:

1. If $x = \frac{-1 - \sqrt{3}\,i}{2}$, $i^2 = -1$, then the numerical value of $x^8 + x^4 + 5 =$ ....

2. If $\lfloor n$, $\lfloor n-2$, n $\lfloor 2-n$ are the side lengths of a triangle, then the numerical value of the perimeter of the triangle = ...

3. If $\overrightarrow{A} = (-2, k, -3)$ is parallel to the straight line $\frac{x+2}{4} = \frac{y}{8} = \frac{z-1}{6}$, then k = ...

4. The measure of the angle which the vector $\overrightarrow{A} = (3, 4, \sqrt{11})$ makes with the positive direction of x-axis equals ....

5. If the two planes $x - 3y + mz = 5$, and $3x + ky + 6z = 10$ are parallel, then $k \times m =$ ....

6. The distance between the two parallel planes $4x + 6y + 12z + 18 = 0$ and $4x + 6y + 12z - 10$ = ...

Second question: Choose the correct answer:

1. $1 - 6x + \frac{6 \times 5}{2 \times 1} x^2 - \frac{6 \times 5 \times 4}{3 \times 2 \times 1} x^3 +$ .... $+ x^6 = 64$ then x =

   a) -1 b) 3 c) {-1, 3} d) 2

2. $(\frac{5 - 3\omega^2}{5\omega - 3} - \frac{2 - 7\omega}{2\omega^2 - 7})^2 =$

   a) 3 b) -3 c) 3i d) -3i

3. If the two straight lines: $\frac{x+1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$, $\frac{x}{3} = \frac{y+1}{4} = \frac{z-1}{k}$ are perpendicular , then k =

a 4   b -4   c $\frac{9}{2}$ d $-\frac{9}{2}$

4. The equation of the sphere whose centre is (3, -2, 1) and its radius length equals = 5 cm is

a $(x+3)^2 + (y-2)^2 + (z+1)^2 = 5$ b $(x+3)^2 + (y-2)^2 + (z+1)^2 = 25$

c $(x-3)^2 + (y+2)^2 + (z-1)^2 = 25$ d $(x-3)^2 + (y+2)^2 + (z-1)^2 = \sqrt{5}$

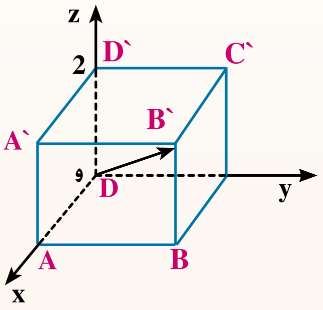
5. The measure of the angle included between the two planes $x + \sqrt{2}\, y - z = 5$, $x - \sqrt{2}\, y + z = 1$ equals

a 0°   b 45°   c 90°   d 135°

6. **In the opposite figure** , A B C D A\`B\`C\`D\` is a cuboid A (4, 0, 0) , C (0, 9, 0) D\` (0, 0, 7) , then $\| \overrightarrow{AC`} \|$ =

a $\sqrt{146}$ b $\sqrt{114}$

c 5   d $\sqrt{20}$



Answer the following questions

Third question:

1. In the expansion of (2x -3)15 according to the descending powers of x, find the values of x which makes 13T3 + 10T4 + T5 = 0

2. Without expanding the determinant , prove that $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = 2 \begin{vmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix}$

Fourth question:

1. Prove that : $\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i \sin n\left(\frac{\pi}{2} - \theta\right)$

2. Find the equation of the straight line passing through the point (3 , -1 , 0) and intersects the straight line $\vec{r} = (2, 1, 1) + t\,(1, 2, -1)$ orthogonally

Fifth question:

1. Use the multiplicative inverse of the matrix to solve the set of following equations:

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, $\frac{1}{x} - \frac{1}{y} + \frac{2}{z} = \frac{1}{2}$, $\frac{2}{x} + \frac{3}{y} - \frac{4}{z} = \frac{4}{3}$

where x , y and z are not equal to zero

2. Find the component of $\overrightarrow{AB}$ in the direction of $\vec{m}$ where A (2 , 1 , 0) , B (3, 1, $\sqrt{3}$)

$\vec{m} = (3, 2, 2\sqrt{3})$

Answers of exercises

Unit 1 : Permutations, combinations and Binomial theorem

Answers of exercises (1 - 1)

(1) c (2) d (3) b (4) d

(5) a (6) c (7) a (8) b

(9) b (10) 40 (11) 14 (12) 2520

(13) a 2401 b 840

(14) a 24 b 48 c 40

(15) a $n = 2 \quad r = 8$ b $n = 7 \quad r = 3$

c $n = 7 \quad r = 4$ d $r = 2 \quad n = 5$

(16) $n = 20$, $r = 13$ (17) $\frac{16}{7}$ (18) $n = 16$

(19) $r = 4$, $n = 7$ (20) The least value of the variable $n = 5$

(21) $r = 2$, $n = 11$ (22) $\frac{28}{9}$ (23) $r = 4$

(24) $x \in \{0, 1, 2, 3, .., 30\}$, $y = \frac{1}{2}$, $y = \frac{-1}{2}$

(25) $n \in \{8, 9, 10, ...\}$ (26) $m + n = 20$

(27) a $n = 2$ b $n = 2$ c $n = 6$

(28) $r = 8$, $n = 79$

(29) a $n = 13$ or $n = 12$ b $r = 5$, $r = 9$

(30) a $r = 4$, $n = 14$ b $r = 4$, $n = 12$

c $r = 2$ then $n = 10$

d $n = 13$ e $r = 10$, $n = 10$

(31) 6 (32) 10 (33) 218025

(34) 1260 (35) 304

(36) a 4 b 10 c 20

(37) a 9 b 20 c 54

(38) a 495 b 252 c 369

Answers of exercises (1 - 2)

(1) a (2) d (3) d (4) c

(5) b (6) d (7) b (8) c

(9) a (10) $x = 1 \quad x = -3$

(11) a 1,0151 b 0,986 c 2,003

d 0,321

(12) $x = \frac{1}{2}$ (13) Proof

(14) a $16x^{-4} + 16x^{-2} + 6 + x^2 + \frac{1}{16}x^4$

b $x^5 - 5x^3 + 10x - 10x^{-1} + 5x^{-3} - x^{-5}$

c $2x^4 + 24x^2 + 8$

d $180x + 480x^3 + 64x^5$

(15) $n = 8 \quad x = \pm 2$ (16) $n = 14$

(17) $2ab = 1$ (18) $t_7 = 924x^6$

(19) $t_6 = -231x^7$ $t_7 = 924x^4$

(20) $84x$ (21) $x = \frac{2}{3}$ (22) $\frac{t_6}{t_5} = \frac{1}{20}$

(23) $x = \pm\frac{8}{15}$

Answers of exercises (1 - 3)

(1) c (2) c (3) c (4) c

(5) c (6) c (7) d (8) b

(9) c (10) c

(11) $t_9 = 495$ (12) $\frac{1365}{128}$

(13) There is no term including x^{-6}

(14) 4608 - 5376 the expansion doesnot contain x^2

(15) $n = 15$ (16) 1920 (17) $^{2n}g_n$

(18) the term free of $x = {}^{16}g_8$

(19) when $r = 3$ we find $k = 1$ when $r = 4$ we find $k = 2$

when $r = 5$ we find $k = 5$ $\frac{3}{10}$

(20) $a = 2$ (21) $a = \pm\frac{\sqrt{3}}{3}$ (22) $x = \pm\frac{\sqrt{2}}{4}$

(23) t_4 is the term free of $x = 84$

the two middle terms are t_5 , t_6 $x = -1$

(24) $x = \frac{1}{3}$ (25) $\frac{21}{55}$

Answers of exercises (1 - 4)

(1) b (2) c (3) d (4) b

(5) a $\frac{15}{2x^4}$ b $\frac{x^4}{3}$ c $\frac{28x^8}{45}$ d $\frac{10}{63}$

6) $\frac{4}{11}$ 7) n = 5 a = 2 , b = 3

8) n = 5

9) n = 8

10) t_4, t_5 are equal and each has the greatest numerical value in the expansion

11) n = 8

Answers of general exercises

1) a 2) c 3) d 4) c

5) a 6) c 7) d 8) b

9) c 10) c 11) b

12) r = 3 n = 6 13) n = 10 m = 4

14) r = 4 n = 10 15) n = 10 r = 1

16) r = 7 n = 11 17) r = 4 n = 10

18) a 1530 b 111 19) n = 6 r = 5 , 35

20) r = 8 n = 79 21) r = 5 least values n = 5

22) By differentiating with respect to x

by substituting in x = 1 $n \times 2^{n-1}$

23) a by substituting in x = 1 in both sides

b by substituting in x = -1 0

c by substituting in x = 3 4^{10}

24) n = 15 r = 10 $t_{11} = 3003$

25) n = 7 26) n = 29 $\lfloor 10$ = 3628800

27) n = 9 28) terms are t_{20}, t_{21}, t_{22} $^{27}g_9$

29) n = 10 , m = $\frac{-1}{3}$ $t_6 = 252$

30) t_{15} is the term free of x

g14 = $\frac{t_{12}}{t_{11}} = \frac{3}{2}$

31) x = $\frac{1}{2}$ there is no term free of x

32) terms are t_5 , t_6, t_7 33) x = $\frac{3}{5}$

34) 495 35) x = $\frac{3}{2}$ 36) $a + b x^2 = 0$

37) n = 9 38) x = $\pm \frac{1}{2}$

39) n = 10, x = $\frac{5}{6}$ 40) n = 12, c = 1

41) n = 8 x = ± 1 , y = ± 2

42) (1) $924 x^{-6}$ (2) t4 = $^{12}g_3 (\frac{2}{3})^6 x^3$

43) 40 44) n = 20 , x = $\sqrt[3]{2}$

the expansion doesnot contain a term free of x

45) r = 4 r = 3 c = $\frac{2}{3}$

46) r = 10 $t_{11} = {}^{15}g_{10}$ $\frac{21}{80}$

47) a r = 6 k = 3 r = 7 k = 7 b $\frac{4}{35}$

48) proof 49) r = 2 n = 6 y = $\frac{1}{\sqrt[3]{2}}$

50) a = $\pm \frac{\sqrt{10}}{2}$ 51) $\frac{t_7}{t_6} = \frac{3}{2}$

52) $\frac{t_6}{t_5} = \frac{1}{27x^3}$ when x = $\frac{1}{3}$ $t_6 = t_5$

53) r = 6 x = $\frac{3}{2}$ 54) $\frac{21}{55}$

Answer of accumulative test

1) a 2) c 3) b 4) c

5) c 6) n = 25 x = $\pm \frac{1}{4}$

7) a n = 10 b r = 2 $t_3 = 45$

8) n = 25 m = $\frac{1}{100}$ 9) first : 105 second: 61

10) n = 16 x = $\pm \frac{4}{3}$

Unit 2 : Complex number

Answers of exercises (2 - 1)

1) (3, -4) 2) x

3) 5 4) 1 5) -5 6) 1

7) $\sqrt{2}\, e^{\frac{3\pi}{4}i}$ 8) 120°

9) 4 (cos (-6°) + i sin (-60°))

10) θ 11) 60° 12) π

13) $L_1 L_2$ i 14) 180°

15) 2 16) $e^{\frac{3\pi}{4}i}$ 17) 300°

18) 1 19) 3(cos 120° + i sin 120°)

20) - θ

21) a 5 (cos $\frac{\pi}{2}$ + i sin $\frac{\pi}{2}$)

b 4 (cos (- $\frac{5\pi}{6}$) + i sin (- $\frac{5\pi}{6}$))

c $\sqrt{2}$ (cos (- $\frac{\pi}{4}$) + i sin (- $\frac{\pi}{4}$))

d 5 (cos 126,9 °+ i sin 126,9°)

e 4 (cos (-40) + i sin (-40))

22 a $\sqrt{2}$, $-\frac{\pi}{4}$ b 2 , $\frac{\pi}{6}$

c 2 , $-\frac{3\pi}{4}$ d sec20° , $\frac{\pi}{9}$

23 i 24 $\frac{1}{2}e^{\frac{\pi}{3}i}$ 25 $\frac{1}{2}e^{\frac{-\pi}{3}i}$

26 a $\frac{1}{2}+\frac{\sqrt{3}}{2}$ i b $-\sqrt{2}+\sqrt{2}$ i

c $\frac{3\sqrt{2}}{2}-\frac{3}{2}$ i

27 $\frac{1}{2}e^{\frac{5\pi}{3}i}$ 28 - 64

29 |z| = 1 30 $-\sqrt{3}$ (cos 45 + i sin 45)

31 a $\frac{\pi}{2}$ b $\frac{13}{12}\pi$ c $\frac{11}{21}\pi$ d π

32 sin = $\frac{-i}{2}$ ($e^{\theta i}$ - $e^{-\theta i}$)

Answers of exercises (2 - 2)

1 a 8 $\cos^4 8 - \theta \cos^2 1 + \theta$

b 16 $\sin^5 20 - \theta \sin^3 5 + \theta \sin\theta$

2 a 2 2i -2 - 2 i

b $z_1 = 1 + \sqrt{3}$ i $z_2 = -2$

$z_3 = 1 - \sqrt{3}$ i

c $z_1 = 2$ i $z_2 = -\sqrt{3}$ - i

$z_3 = \sqrt{3}$ - i

3 z_1 = 3 (cos 36 °+ i sin 36°)

z_2 = 3 (cos 108° + i sin 108°)

z_3 = 300

z_4 = 3 (cos (-108°) + i sin (-108°))

z_5 = 3 (cos (-36 °) + i sin (-36°))

4 $z_1 = \sqrt{2}\ e^{\frac{\pi}{12}i}$ $z_2 = \sqrt{2}\ e^{\frac{7\pi}{12}i}$

$z_3 = \sqrt{2}\ e^{-\frac{11\pi}{12}i}$ $z_4 = \sqrt{2}\ e^{-\frac{-5\pi}{12}i}$

5 a $z_1 = \sqrt{3}$ - i $z_2 = -\sqrt{3}$ + i

b z1 = $\sqrt[4]{2}$ (cos ($-\frac{\pi}{8}$) + i sin ($-\frac{\pi}{8}$))

$z_2 = \sqrt[4]{2}$ (cos ($-\frac{\pi 7}{8}$) + i sin ($-\frac{\pi 7}{8}$))

c z_1 = 2 + 2 i z_2 = -2 - 2 i

d 4 + i , -2 - i e ± (3 - 2 i)

6 z_1 = 2 z_2 = 2 (cos 120 + i sin 120)

z_3 = 2 (cos (120°) + i sin (-120°))2

7 $z_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$z_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$z_3 = \cos(-\frac{3\pi}{4}) + i\sin(-\frac{3\pi}{4})$

$z_4 = \cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})$

8 when k =0 magnitude = 2 + 2 i

when k =1 magnitude = -2 - 2i

9 $z_1 = 2e^{\frac{\pi}{8}i}$ $z_2 = 2e^{-\pi\frac{7}{8}i}$

10 18 - 26 i , -18 + 26 i

11 $\cos^4\theta = \frac{1}{8}$ [cos 4 4 + θ cos 2 3 + θ]

Answers of exercises (2 - 3)

1 - 9 $^2\omega$ 2 9 3 1 4 -1

5 4 6 -2 7 - 37 8 -1

9 $^2\omega$ 10 2 11 (أ - b)2

12 1 13 ± $\sqrt{3}$ i

14 (1,1) 15 3 16 - $^2\omega$

17 1 18 6

20 a 2 b -1 c -1 d $\frac{-27}{49}$

e i

21 0 22 $x^2 - x + 1 = 0$

23 -2 i algebraic

z = 2 (cos ($-\frac{\pi}{2}$) + i sin ($-\frac{\pi}{2}$))

trigonometric

$2e^{-\frac{\pi}{2}i}$ exponential

$\sqrt{2}$ (cos ($-\frac{\pi}{4}$) + i sin ($-\frac{\pi}{4}$))

$\sqrt{2}$ (cos ($-\frac{3\pi}{4}$) + i sin ($-\frac{3\pi}{4}$))

24 n = 3 k where k∈ y

25 a 1 +ω b 3

Answers of general exercises

1 1 2 2 (cos 120 °+ i sin 120°)

3 90 - θ ° 4 - i + ω

5 -1 6 60° 7 a 8 b

9 d 10 b 11 c 12 c

13 c 14 b 15 1 16 b

17 b

18 a -1 b $3-3i$ c $2+2i$

19 a $z_1 = 3\sqrt{3}\ (\cos 150 + i \sin 150)$

b $-z_2 = \sqrt{3}\ (\cos 105 + i \sin 105)$

c $z_1 z_2 = 9\ (\cos(-105) + i \sin(-105))$

d $\frac{z_1}{z_2} = 3\ (\cos 45 + i \sin 45)$

20 a $z_1 = 7\ e\ i$ b $z_2 = 5\ e^{\frac{-\pi}{2}i}$

c $z_3 = \frac{1}{2}\ e^{\frac{\pi}{3}i}$ d $z_4 = \sqrt{7}\ e^{0,71\ i}$

e $z_5 = 4\ e^{\frac{\pi}{6}i}$

21 modulus $z = 2\cos\frac{\theta}{2}$ amplitude $z = \frac{\theta}{2}$

22 $\frac{1}{\sqrt{2}}$

Answer of accumulative test

1 a first b third c fourth

2 3 , 1

3 a $2, \frac{\pi}{6}$ b $\sqrt{2}, \frac{3\pi}{4}$

c $4, \frac{\pi}{2}$ d $5, \pi$ e $5, -0{,}93d$

4 i $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$\cos(\frac{-3\pi}{4}) + i\sin(\frac{-3\pi}{4})$

5 $-5 + 2i$

6 $6e^{\frac{\pi}{3}i}$ $2(\cos\frac{\pi}{9} + i\sin -\frac{\pi}{9})$

$(\cos\frac{7\pi}{9} + i\sin -\frac{7\pi}{9})$

$2(\cos\frac{-5\pi}{9} + i\sin -\frac{-5\pi}{9})$

7 $\frac{1}{2}(-1 - \sqrt{3}\ i)$

$(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3}))$

$(\cos(\frac{2\pi}{3}) + i\sin -\frac{2\pi}{3})$

8 $\sqrt{6}\ e^{\frac{\pi}{12}i}$ $z = \sqrt[8]{6}\ (\cos(\frac{\pi}{12} + 2k\pi)$

$+ i\sin(\frac{\pi}{12} + 2k\pi))$

where $k = 0, 1, 2, 3$

9 $2\ e^{\frac{-2\pi}{3}i}$

10 $\left|\frac{1+z}{1-z}\right| = \cot\frac{\theta}{2}$ amplitude $= \frac{\pi}{2}$

11 $z = \omega$

$z_1 = \frac{\sqrt{3}}{2}\ (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

the two roots are $\frac{1}{\sqrt[4]{3}}\ (\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$\frac{1}{\sqrt[4]{3}}\ (\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4})$

unit 3: Determinants and Matrices

Answers of exercises (3 - 1)

1 a **2** b **3** a **4** a

5 a **6** b **7** d **8** d

9 d **10** c **11** -420

12 -2 **13** $x = -1$

14 $x_1 = \frac{3+\sqrt{5}}{2}$, $x_2 = \frac{3-\sqrt{5}}{2}$

15 $x = 1$

16 0 **17** 0

Answers of exercises (3 - 2)

1 b **2** a **3** c **4** c

5 a 5 -4 b 1

c 3, $x = \frac{-23}{5}$

6 a $\frac{1}{2}\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}$ b $\frac{1}{9}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

c $\frac{1}{1}\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$

d $\frac{1}{1}\begin{pmatrix} \sec\theta & -\tan^2\theta \\ -1 & \sec\theta \end{pmatrix}$

e $a^{-1} = \begin{pmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}$ و $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Answers of exercises (3 - 3)

1 d **2** a **3** b **4** a

5 a **6** c **7** b **8** b

9 a **10** b

11 a $x = 8$, $y = -12$

b $a = 2$, $b = 1$ c $c = 5$, $d = 2$

d $b = 7$, $c = 0$, $d = -1$

e $x = 1$, $y = -1$, $z = 2$

f $x = 1$, $y = 2$, $z = 3$

12 a $x = \frac{-13}{3}$ $y = \frac{3}{3} = 1$ $z = \frac{23}{3}$

b $x = 1$ $y = 1$ $z = -2$

c $x = -2$ $y = 1$ $z = 1$

d $x = 2$ $y = 1$ $z = 1$

13 theoretical proof

14 a $x = L, y = L, z = -L$

b $x = -2L, y = z = L$

c $x = L, y = L, z = -L$

Answers of general exercises

1 0 **2** $\{10\}$ **3** 0 **4** 0

5 ± 3 **6** $\begin{pmatrix} 4 & -2 \\ 10 & 5 \end{pmatrix}$ **7** 1

14 $k = -1$ **15** $k = -4$

16 $(2, -1, 3)$ **17** $(-4, 3, 2)$

18 General solution = $\{(-2k, k, 0)\}$

19 $(1, -2, 2)$ **20** $(1, -1, 2)$

21 Equations have no solutions

Answer of accumulative test

1 70 **2** $\begin{pmatrix} -17 & 4 \\ 6 & -1 \end{pmatrix}$ **3** ± 4

4 7 **5** 0 **6** $t - \{13, -1\}$

7 1 **8** 3 **10** $r(a) = 2$

12 $(1, 2, 3)$

14 Equations have a unique solution

16 a a non-singular b a singular

c a non-singular d a singular

Second: Analytic solid geometry

Unit 1 : Geometry and Measurement in two and three dimensions

Answers of exercises (1 - 1)

1 0 **2** $x\,z$, $y = 0$

3 $(6,4,0)$, $(6, 0, 2)$ **4** $(\frac{1}{2}, -2, 3)$

5 $(x-2)^2 + (y+1)^2 + (z-4)^2 = 25$

6 1 **7** 5 **8** $(1, \frac{3}{2}, 6)$

9 $x^2 + y^2 + z^2 = 25$

10 $(x-2)^2 + (y+3)^2 + (z-4)^2 = 16$

11 a $2\sqrt{13}$ b $\sqrt{13}$ c 5

12 a $3\sqrt{5}$ square unit b $2\sqrt{21}$ square unit

13 $(3, 0, 0), (3, 3, 0), (3, 0, 3)$

$(0, 3, 0), (0, 3, 3), ((0, 0, 3)$

$(3, 3, 3), (0, 0, 0)$

14 $3 \pm 2\sqrt{6}$

15 a $(\frac{5}{2}, \frac{-1}{2}, \frac{3}{2})$ b $(\frac{-9}{2}, \frac{9}{2}, \frac{13}{2})$

16 $a(-6, 10, -1)$

17 a $(x-3)^2 + (y+1)^2 + (z-2)^2 = 7$

b $(\frac{3}{2}, 1, -2)$

$(x - \frac{3}{2})^2 + (y-3)^2 + (z+1)^2 = \frac{29}{4}$

c $(x-1)^2 + (y+6)^2 + (z-1)^2 = 42$

18 a center $= (0, 0, 0)$, $r = 3$

b center $= (1, -2, 0)$, $r = \sqrt{5}$

c center $= (\frac{1}{2}, \frac{3}{2}, 1)$, $r = 1$

19 $(x-3)^2 + (y-3)^2 + (z-3)^2 = 9$

20 $(1, 0, 2)$ **21** 4 length unit

22 $z = 2$ **23** Zyad's answer

Answers of exercises (1 - 2)

1 $\sqrt{29}$ **2** $\hat{i} - \hat{j} + 4\hat{k}$

3 $(\frac{4}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{2}{\sqrt{29}})$

4 41' 57"36° **5** 90°

6 ± 2 **7** 82,355°

8 $-\hat{i} - 6\hat{j} + \hat{k}$ **9** $(\frac{-2}{3}, \frac{1}{3}, \frac{2}{3})$

10 a $(6, -5, 1)$ b $(8, -9, 2)$

c $(2, -3, 2)$

11 a $(4, -2, 8)$ b $(-3, \frac{-13}{2}, \frac{17}{2})$

c $(-2, -19- 27)$

12 a $\sqrt{5}$ b 3 c 1 d $\sqrt{17}$

13 proof **14** $(-14, 7, 14)$

15 $\vec{A} = (0, y, z)$

16 $\|\vec{A} + \vec{B}\| \leqslant \|\vec{A} + \vec{B}\|$

(17) $\pm \frac{5\sqrt{3}}{3}(\hat{i} + \hat{j} + \hat{k})$

Answers of exercises (1 - 3)

(1) 0 (2) $\hat{i}$ (3) 2 (4) orthogonal

(5) parallel (6) 142,125°

(7) 32 (8) $\hat{k}$ (9) -7 (10) [-1, 1]

(11) 57,02° (12) 3

(13) a 10 b -28 c 0

(14) a 32,51° b 98,7° c 90°

(15) $-15\hat{i} - 7\hat{j} - 9\hat{k}$ b $10\hat{i} - 5\hat{j} - 3\hat{k}$

c $24\sqrt{3}\ \overrightarrow{C}$

(16) a 144 b $144\ \overrightarrow{C}$ c 144

d $-288\ \overrightarrow{C}$ e 0 f $144\ \overrightarrow{C}$

(17) $\left(\frac{7}{\sqrt{75}}, \frac{5}{\sqrt{75}}, \frac{1}{\sqrt{75}}\right)$

(18) a 16,7183 units b $\frac{5}{2}\sqrt{19}$

(19) a 5 b $\sqrt{80}$

(20) 9 units

(21) a $\overrightarrow{A}$, $\overrightarrow{B}$ are not orthogonal

$\overrightarrow{A}$, $\overrightarrow{B}$ are not parallel

b $\overrightarrow{E}$, $\overrightarrow{O}$ are not orthogonal

$\overrightarrow{E}$, $\overrightarrow{O}$ are not parallel

c $\overrightarrow{A}$, $\overrightarrow{B}$ are not orthogonal

$\overrightarrow{A}$, $\overrightarrow{B}$ are parallel

Answers of general exercises

(1) x z , y = 0

(2) a (6 , 8 , 0) b (0 , 8 , 0)

c 53,13°, 36,87°, 90°

(3) (5 , -3, 2) (4) 7 or -1

(5) $\frac{-2}{3}$, -12 (6) 45°

(7) y = 0 (8) $x^2 + y^2 + z^2 = 14$

(9) $(4, 1, -\frac{1}{2})$ (10) 0

(11) $(\frac{3}{5}, \frac{-4}{5}, 0)$ (12) $\ell^2 + m^2 + n^2 = 1$

(13) 0 (14) y -axis

(15) proof (16) (0 , 0, 3) 3

(17) (3, 1, -7) (18) $(\frac{1}{3}, \frac{-2}{2}, \frac{2}{3})$

(19) 31,96° $6,5\hat{i} + 2,2568\hat{j} + 11,0292\hat{k}$

(20) $\sqrt{14}$

(21) $\|\overrightarrow{A}\| = \|\overrightarrow{B}\| = 1$ $\theta = 90°$

(22) a 36 b -36 c 0

(23) -23 (24) - 400 joule

(25) a proof b proof

Answer of accumulative test

(1) 4 (2) -3 (3) $(2, \frac{-1}{2}, 1)$

(4) a (0, 8, 4) b (0, 0, 4)

(5) $(x-1)^2 + (y+3)^2 + (z+1)^2 = 13$

(6) (-3 , -3, 13) (7) $\left(\frac{5}{\sqrt{38}}, -\frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}\right)$

(8) ±3 (9) 5 (10) -3

(11) $\frac{18}{5}$ (12) 90° (13) 9 square unit

(14) $x^2 + (y-4)^2 + z^2 = 16$

(15) $\sqrt{14}$ $\sqrt{27}$

(16) $\pm 21(\hat{i} + \hat{j} + \hat{k})$

(17) (-24 , 48, 36)

(18) a (1 , -11, -5) b (6 , -66 , -30)

c (-2 , 22, 10)

(19) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(20) $\theta x = 64,896°$ $\theta y = 124,45°$

$\theta z = 45°$

Unit 2 : Straight Lines and planes in space

Answers of exercises (2 - 1)

(1) $\overrightarrow{r} = (2, -1, 3) + k(-1, 4, 2)$

(2) 90° (3) 60° (4) $\frac{1}{\sqrt{11}}$

(5) (2 , -2 , 1)

(6) a $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

b $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

(7) a x = 4 + 2k , y = -2 + k , z = 5 -k

$\frac{x-4}{2} + \frac{y+2}{1} + \frac{z-5}{-1}$

b $x = 3 + 2k$, $y = -1 - k$, $z = 5 + k$

$\frac{x-3}{2} = \frac{y+1}{-1} = \frac{z-5}{1}$

c $x = 3 + 3k$, $y = -2 - 6k$, $z = -k$

$\frac{x-3}{3} = \frac{y+2}{-6} = \frac{z}{-1}$

d $x = 3 + k$, $y = 2 + k$, $z = 5 + k$

$x - 3 = y - 2 = z - 5$

8 $\vec{r} = (3, -2, 2) + k(1, 4, -3)$

9 a $\vec{r} = (1, -2, 1) + k(-1, 1, -2)$

b $\vec{r} = (8, 1, 4) + k(4, 1, 1)$

c $\vec{r} = (3, 1, -2) + k2(-19, -11, 4)$

10 a 60° 30 ` 41`` b 53°41 ` 23``

c 84° 2 ` 20``

11 a $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

b $C_1 C_2 + b_1 b_2 + c_1 c_2 = 0$

12 $\vec{r} = (1, -1, 0) + k(5, -1, -1)$

13 $n = 7$, $(\frac{23}{5}, \frac{-3}{5}, \frac{41}{5})$

Answers of exercises (2 - 2)

1 c   2 c   3 c

4 b   5 c   6 b

7 $2x - 3y + 4z = 21$

a doesnot lie   b is not parallel to the plane $\vec{C}$

8 a (3, 0, 0)   b (-1, -2, 0)

c (2, 1, 3) , (-1, 2, 1) , (-4, 3, 0)

d (1, 1, 1,) , (0, 2, 2) , (2, 0, 0)

9 $x + 2y - 3z = 0$

10 $4x + 10y - 7z = -2$

11 $-4x - 10y + 7z = 2$

14 a $2x + 3y + 5z = 27$

b $-2x + 4y - z + 4 = 0$

c $-x + 3y + 2z = 9$

15 (4, 2, 3)

16 (10, 5, 4) . $\vec{r} = 20$ directed form

17 a $z = 0$ b $z = 2$

c $y = 0$, $y = 5$, $x = 0$

18 $9x + 17y + 16z + 23 = 0$

19 a ∴ $78,578 = \theta°$ b $63,07 = \theta°$

c $59,53 = \theta°$

20 a = (-1, 2, 1)   b $2\sqrt{6}$

d the direction equation to the intersection line

$\vec{r} = (0, \frac{-1}{19}, \frac{-17}{19}) + k(19, -9, 37)$

21 a $3y + 4z - 20 = 0$

b $x + 2y + 2z - 12 = 0$

c 5 = ف   2 = ط

d the direction equation to the intersection line

$\vec{r} = (2, 0, 5) + k(-2, 4, -3)$

e $X = -12$ $X = 3$

Answers of general exercises

1 b   2 a   3 c   4 c

5 c   6 c   7 c   8 b

9 d   10 d   11 $\frac{\sqrt{370}}{10}$

13 point (1, -2, 7)   14 (2, -1, 2)

15 (-2, 4, 2) length = $\frac{21}{6}$ 16 perpendicular

Answer of accumulative test

1 60°   2 $\frac{\sqrt{30}}{6}$

3 $x = -1 + 2k$, $y = -k$, $z = 3 - 3k$

4 60 °   5 $5x + 2y - 3z - 19 = 0$

6 2,5   7 (-1, 2, 0)   8 c

9 c   10 a   11 c

12 c

13 a $\frac{x-1}{5} = \frac{y-3}{4} = \frac{z-9}{2}$

b $\frac{x}{3} = \frac{y-2}{-1} = \frac{z}{4}$

14 a 50,1°   b 45°

(15) $10x - 22y + 19z = 49$

(16) 57,456

Answer of tests

Test 1

Q 1

(1) d (2) a (3) d (4) b

(5) a (6) d (7) $2x - 3y + 4z = 21$

Q 2

(1) -6048 (2) {2}

(3) 6 (4) (8, -5, -6)

(5) $(x-2)2 + (y+3)2 + (z-1)2 = 20$

(6) $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-4}{2}$

Q 3

(1) the term free of x is t6 = 3075072

(2) $x = -3 + 12k$, $y = \frac{1}{2} + 15k$

$z = -\frac{2}{3} + 8k$

Q 4

(1)

$$\begin{pmatrix} 68 & -31 & -5 \\ 41 & -19 & -3 \\ -13 & 6 & 1 \end{pmatrix}$$

(2) $z_1^{\frac{1}{2}} = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$z_2^{\frac{1}{2}} = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$

Q5

(1) $(x, y, z) = (1, 2, 3)$

(2) $(x, y, z) = \left(2, -\frac{5}{2}, \frac{5}{2}\right)$

Test 2

Q1

(1) c (2) d (3) a (4) a

(5) b (6) a

Q2

(1) ω (2) 0 (3) $\frac{8}{3}$

(4) 0 or $\frac{1}{2}$ (5) -2 (6) -33

Q3

(1) 220 (2) $e = 30°$

Q4

(1) r(3 = (أ = n (x, y, z) = (2, -1, 1)

(2) $= \frac{1}{2}\left(\cos\left(-\frac{\pi}{2}i\right) + i\sin\left(-\frac{\pi}{2}i\right)\right)$

$2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

$\sqrt{2}\left(\cos\left(-\frac{\pi 3}{4}\right) + i\sin\left(-\frac{\pi 3}{4}\right)\right)$

Q5

(2) 10

Test 3

Q1

(1) c (2) b (3) a (4) d

(5) b (6) a

Q2

(1) $-\frac{\pi}{6}$ (2) 1 (3) 3 (4) 5

(5) -10,5 (6) 12

Q3

(1) $n = 6$, $m = 3$, $a = 243$

$(x, y, z) = (k, -k, -k)$

Q4

(1) $\left(\frac{\sqrt{2}}{2} - 1\right) - \frac{\sqrt{2}}{2} i$

(2) Perpendicular length = 0

Q5

(2) (-4, -8, -6) , (-4, 8, -6) , -12

Test 4

(1) b (2) a (3) d (4) c

(5) a (6) c

Q2

(1) -40 (2) 2 (3) (-4, 6, -1)

(4) 100 (5) $\left(\frac{2}{5}, \frac{3}{5}, \frac{2\sqrt{3}}{5}\right)$

(6) $\sqrt{10}$

Q3

(1) $t3^6$ is the greatest term and its value is g_2

$(3)^4\ (2)^2 = 4860$

(2) 16 unit 3

Q4

(1) $z_1 = \sqrt{2}\ (\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$z_2 = \sqrt{2}\ (\cos\frac{\pi 3}{4} + i\sin\frac{\pi 3}{4})$

$z_3 = \sqrt{2}\ (\cos\frac{\pi 3-}{4} + i\sin\frac{\pi 3-}{4})$

$z = \sqrt{2}\ (\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$

(2) $a = \sqrt{14}$, $b - \frac{\sqrt{2}}{10}$ $(4, 5, 3)$

Q5

(2) $\cos(-\frac{7}{18}\pi) + i\sin(-\frac{7}{18}\pi)$

first root $= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

second root $= \cos\frac{\pi 5}{6} + i\sin\frac{\pi 5}{6}$

third root $= \cos(-\frac{\pi 5}{6}) + i\sin(-\frac{\pi 5}{6})$

Test 5

Q1

(1) b (2) d (3) a (4) b

(5) d (6) c

Q2

(1) 133 (2) 20 (3) $\frac{1}{9}$ (4) 3

(5) $(x-3)2 + (y-4)2 + (z+5)\ 2 = 9$

(6) $\vec{r} = (2, -1, 4) + k\ (4, 7, 1)$

Q3

(1) $r = 6$ (2) $k = 7$, $k = -1$

Q4

(1) $(x, y, z) = (\frac{1}{3}, \frac{10}{3}, -3)$

(2) $z_1 2 = {}^{\frac{1}{3}}\ e^{-\frac{\pi}{6}i}$ $z_2 2 = {}^{\frac{1}{3}}\ e^{-\frac{\pi}{2}i}$

$z_3 2 = {}^{\frac{1}{3}}\ e^{-\pi\frac{5}{6}i}$

Q5

(2) 11π

Test 6

Q1

(1) c (2) a (3) c (4) c

(5) d (6) b

Q2

(1) 243 (2) 3 (3) (3, 0, 2)

(4) 1 or $-\frac{13}{5}$ (5) -5 (6) -41

Q3

(1) $n = 19$ or $n = 8$

( ) $(x-1)^2 + (y-2)^2 + (x-1)^2 = 3$

Q4

(1) $(x, y, z) = (2, 1, 1)$

(2) $z^{\frac{1}{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$z^{\frac{1}{2}} = \cos(-\frac{\pi 3}{4}) + i\sin(-\frac{\pi 3}{4})$

Q5

(2) $\frac{x-2}{5} = \frac{y-1}{2} = \frac{z+3}{3}$

Test 7

Q1

(1) b (2) c (3) a (4) c

(5) d (6) a

Q2

(1) 1 (2) 3 (3) 45° (4) 8

(5) -1 (6) $\sqrt{157}$

Q3

(1) $\sqrt{z} = e - {}^{\frac{\pi}{36}i}$ $\sqrt{z} = e - \pi^{\frac{35}{36}i}$

(2) $x = 125$

Q4

(1) $n = 18$, $x = \pm\frac{1}{3}$

Q5

(1) $x = \pm\frac{\sqrt{3}}{2}$ $y = \pm\frac{1}{2}$

$z = \pm\frac{1}{\sqrt{2}}$

(2) the intersection point is (2, 2, 2)

Test 8

Q1

(1) 1 , $\frac{1}{10}$ (2) 5 (3) 90° (4) -16

5 $(x-5)^2 + y^2 + (z+1)^2 = 14$ 6 11 or -1

Q2

1 أ 2 b 3 د 4 أ

5 b 6 أ

Q3

1 x = 1 , y = 2 , y = -1 2 $(2, \frac{-5}{2}, \frac{5}{2})$

Q4

1 $z^{\frac{1}{2}} = \sqrt{2}\ (\cos 0^\circ + i \sin 0^\circ)$

$z^{\frac{1}{2}} = \sqrt{2}\ (\cos \pi + i \sin \pi)$

2 r(a) = 2 r(a) < the number of unknown

Q5

1 $\frac{15}{112}$ 2 k = 10 or k = -4

Test 9

Q1

1 10 2 ±2 3 $\frac{\sqrt{2}}{5}$ 4 3

5 $\frac{\sqrt{3}}{4}$ أو $\frac{-\sqrt{3}}{4}$ 6 9

Q2

1 0 2 3 3 $6, \frac{-2}{3}$ 4 $\pm 2\sqrt{3}$

5 $\frac{-1}{2}$ 6 120°

Q3

1 $z^{\frac{1}{2}} = \cos(-\frac{\pi}{12}) + i \sin(-\frac{\pi}{12})$

$z^{\frac{1}{2}} = \cos(\frac{11}{12}\pi) + i \sin \frac{11\pi}{12}$

2 a = -2

Q4

1 x = 2 , y = 1 , z =1

Q5

1 k = 1 2 $\frac{\sqrt{30}}{2}$

Test 10

Q1

1 4 2 5 3 -4

4 56,44° 5 -18 6 $\frac{9}{7}$

Q2

1 c 2 b 3 d

4 c 5 c 6 a

Q3

1 m.c = $\{0, \frac{1}{2}, \frac{9}{2}\}$

Q4

2 $\vec{r} = (3, -1, 0) + k`(1, -1, -1)$

Q5

1 r = 2

y = 3

z = 6

2 $(\frac{27}{5}, \frac{18}{5}, o\frac{18\sqrt{3}}{5})$